

MATH 10
ASSIGNMENT 21: SUBGROUPS
MAY 6, 2018

Definition. Let G be a group. A subgroup of G is a subset $H \subset G$ which is itself a group, with the same operation as in G . In other words, H must be

1. closed under multiplication: if $h_1, h_2 \in H$, then $h_1 h_2 \in H$
2. contain the group unit e
3. for any element $h \in H$, we have $h^{-1} \in H$.

Examples are given in problem 1 below.

1. Which of the following are subgroups?
 - (a) $G = \mathbb{Z}$ (with operation of addition), $H = 5\mathbb{Z}$ = multiples of 5.
 - (b) $G = \mathbb{Z}$ (with operation of addition), $H = \{n = 5k + 1\}$.
 - (c) $G = S_n$ — permutation group, H = even permutations
 - (d) $G = S_n$ — permutation group, H = odd permutations
 - (e) G = all symmetries of regular n -gon, H = all rotations of regular n -gon
 - (f) G = all symmetries of regular n -gon, H = all reflections of regular n -gon
2. Let \mathbb{Z}_n be the group of all remainders mod n , with operation of addition (it is commonly called the cyclic group of order n). Explain how one can identify this group with the group of all rotations of regular n -gon.
3. Let G be a group, and let $a \in G$. Consider the set of all powers of a :

$$H = \{a^n \mid n \in \mathbb{Z}\} \subset G$$

(note that n can be negative).

- (a) Show that H is a subgroup (this is called the subgroup generated by a). Subgroups of this form are also called *cyclic* subgroups.
- (b) Describe explicitly the cyclic subgroup in \mathbb{Z}_{10} generated by 2; by 3; by 6.
4. Describe all subgroups in \mathbb{Z} (hint: each such subgroup is cyclic).
5. Let $H \subset G$ be a subgroup. For any element $g \in G$, define the subset

$$[g] = gH = \{gh, h \in H\}$$

Subsets of this form are called *cosets*. Note that two different elements can define the same coset.

- (a) List all cosets in the case when $G = \mathbb{Z}, H = 5\mathbb{Z}$.
- (b) Show that two elements x, x' are in the same coset gH iff $x' = xh$ for some $h \in H$.
- (c) Show that two cosets g_1H, g_2H either coincide (if $g_1 = g_2h$ for some $h \in H$) or do not intersect at all.
6. In this problem, we consider the permutation group S_n , with $n \geq 5$
 - (a) Write $(12)(34)$ as a product of cycles of length 3
 - (b) Show that every even permutation can be written as a product of cycles of length 3.