MATH 10
ASSIGNMENT 20: GROUPS
APRIL 22, 2018

Definition 1. A *group* is a set G with a binary operation $*$ and a special element e such that teh following properties hold:

1. Associativity: $(a * b) * c = a * (b * c)$
2. Unit: there is an element $e \in G$ such that for any $g \in G$, we have $e * g = g * e = g$
3. Inverses: for any $g \in G$, there exists an element $h \in G$ such that $g * h = h * g = e$

The operation in groups is also commonly written as a dot (e.g. $g \cdot h$) or without any sign at all (e.g. gh). The unit element is sometimes denoted just 1, and the inverse of g by g^{-1} (see problem 3 below)

A typical example of a group is the group of all permutations of the set $\{1, \ldots, n\}$. It is commonly denoted S_n and called the *symmetric group*. More examples are given in problem 2 below.

1. Let $x, y \in S_9$ be cycles: $x = (1\ 2\ 3\ 4\ 5)$, $y = (5\ 6\ 7\ 8\ 9)$. Compute $xyx^{-1}y^{-1}$ (this is sometimes called the *commutator* of x, y).

2. Show that the following are groups:
(a) Set $\mathbb{Z}$ with the operation of addition
(b) Set $\mathbb{R}$ with the operation of addition
(c) Set $\mathbb{R}^{\times} = \mathbb{R} - \{0\}$ with the operation of multiplication
(d) Set A_n of all even permutations, i.e. permutations with sign $+1$ (it is called the alternating group).
(e) Set of all vectors in 3 dimensional space, with the operation of addition.
(f) Set $\mathbb{Z}_n$ of all integers modulo n with the operation of addition modulo n.
(g) Set O_3 of all rigid motions (i.e., transformations preserving distances) of the 3-dimensional space, with the operation of composition.

3. Prove that in a group, each element g has a *unique* inverse: there is exactly one h such that $gh = hg = e$. (Note that the definition of the group only requires that such an h exists and says nothing about uniqueness). Hint: if h_1, h_2 are different inverses, what is $h_1 g h_2$?

4. Prove that in any group, $(xy)^{-1} = y^{-1}x^{-1}$

5. Consider the set D_n of all symmetries of a regular n-gon (a symmetry is a transformation of the plane that preserves distances and which sends the regular n-gon into itself). Prove that D_n is a group with respect to composition. How many elements are there in D_n? How many of them are rotations?

6. Consider the set R of all rotations of 3-dimensional space which preserve a regular tetrahedron.
(a) How many elements are there in R?
(b) Prove that R is a group.
*(c) Every element of R permutes vertices of the tetrahedron and thus determines an element of S_4. Show that this allows one to identify R with the group A_4 of even permutations of 4 elements.

7. This problem is about permutations of n elements, i.e. about the symmetric group S_n. As we had discussed before, every such permutation can be written as a product of cycles.

(a) How many permutations there are in which the cycle containing element 1 has length 5? length k?

(b) What is the probability that in a randomly chosen permutation s, elements 1 and 2 are in the same cycle?

(c) A theater has 100 seats, all with seat numbers. For todays show, all tickets were sold (and each ticket had a seat number). However, the first 99 people who came to the theater took seats randomly, paying no attention to the seat on their ticket. The last person, however, was a lawyer, so when he came to the theater he insisted upon taking his seat and no other; if his seat was taken, he would ask the person there to move, which forced that person to go to the seat on his ticket, triggering a chain reaction.
What is the probability that everyone will have to move?
What is the probability that the person who came first will have to move?

***8.** (a) How many permutations of set of 100 elements are there that contain a cycle of length 51?

(b) How many permutations of set of 100 elements are there that contain a cycle of length more than 50?

(c) (This is a famous problem, suggested in 2003 by a Danish computer scientist Peter Bro Miltersen. It is a hard problem, but the previous part gives a hint.)
The director of a prison offers 100 death row prisoners, who are numbered from 1 to 100, a last chance. A room contains a cupboard with 100 drawers. The director randomly puts one prisoner's number in each closed drawer. The prisoners enter the room, one after another. Each prisoner may open and look into 50 drawers in any order. The drawers are closed again afterwards. If, during this search, every prisoner finds his number in one of the drawers, all prisoners are pardoned. If just one prisoner does not find his number, all prisoners die. Before the first prisoner enters the room, the prisoners may discuss strategy — but may not communicate once the first prisoner enters to look in the drawers. What is the prisoners' best strategy?
Note: there is no strategy that guarantees the prisoners win, but there are strategies that offer a chance of survival significantly better than $(1/2)^{100}$.