MATH 10
ASSIGNMENT 13: DIMENSION

FEBRUARY 4, 2018

Review of last time

Recall that every system of linear k equations in n variables can be written in the form $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

are vectors in $\mathbb{R}^n$, and A is a matrix (rectangular array of entries) with k rows (one row for ech equation) and n columns. Thus, the system is completely determined by the "augmented matrix" $A \mid \mathbf{b}$.

Using elementary row transformations, such an augmented matrix can be brought to the 'row echelon form", where each row begins with some number of zeroes, and each next row has more zeroes than the previous one:

$$\begin{bmatrix} X & * & * & * & * & * & * & \mid * \\ 0 & 0 & 0 & X & * & * & * & \mid * \\ 0 & 0 & 0 & 0 & X & * & * & \mid * \end{bmatrix}$$

(here X's stand for non-zero entries).

To solve such a system, we do the following:

- Variables corresponding to columns with X's in them are called pivot variables; the remaining ones are called free variables.
- Values for free variables can be chose arbitrarily. Values for pivot variables are then uniquely determined from the equations.

For example, in the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 + 3x_3 &= 6 \end{aligned} \tag{1}$$

varaibles x_1, x_2 are pivot, and variable x_3 is free, so we can solve it by letting $x_3 = t$, and then

$$\begin{aligned} x_2 &= 6 - 3x_3 = 6 - 3t \\ x_1 &= 5 - x_2 - x_3 = -1 + 2t \end{aligned}$$

Systems with no solutions

It could happen that a system of linear equations has no solutions. For example, if the augmented matrix is

$$\begin{bmatrix} 1 & 1 & \mid 2 \\ 0 & 0 & \mid 1 \end{bmatrix}$$

then the second equation reads $0 \cdot x_1 + 0 \cdot x_2 = 1$, which clearly has no solutions. It happens if in the row echelon form, there is an row which has all zero entries except the last one (corresponding to the right hand side of the equation), which is non-zero.

Dimension

If a system of linear equations has solutions and has d free variables, it means that a general solution depends on the choice of d numbers $t_1, \dots, t_d$ — the values of free variables. Moreover, in this case the general solution can be written in the form

$$\mathbf{x} = \mathbf{a} + t_1\mathbf{v}_1 + \cdots + t_d\mathbf{v}_d$$

for some vectors $\mathbf{a}, \mathbf{v}_1, \dots, \mathbf{v}_d \in \mathbb{R}^n$.

In this case we say that the set of solutions has *dimension d*. It can be shown that the dimension does not depend on how we brought the matrix to row echelon form.

For example, in the system (1) above, the general solution is

$$\mathbf{x} = \begin{bmatrix} -1+2t \\ 6-3t \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

so the space of solutions has dimension 1.

Theorem. *If a system of linear equations has solutions, then the dimension of the space of solutions is given by*

$$d = (\textit{number of variables}) - (\textit{number of nonzero rows in row echelon form})$$

Indeed, the number of free variables is the number of all variables minus the number of pivot variables.

Thus, typically we expect that a system with n variables and k equations has $n-k$ dimensional space of solutions. This is not always true: it could happen that after bringing it to row echelon form, some rows become zero, or that the system has no solutions at all — but these situations are unusual (at least if $k \le n$).

Lines and planes in $\mathbb{R}^3$

If we have a single linear equation $ax_1 + bx_2 + cx_3 = d$ in $\mathbb{R}^3$ (we assume that at least one of a, b, c is non-zero), then, by above, the space of solutions is 2-dimensional, i.e. a general solution can be written in the form

$$\mathbf{x} = \mathbf{a} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2,$$

for some vectors $\mathbf{a}, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$. Such a set is called a *plane*.

If we have a system of two linear equations, then typically we expect it to have a one-dimensional space of solutions, i.e.

$$\mathbf{x} = \mathbf{a} + t\mathbf{v},$$

for some vectors $\mathbf{a}, \mathbf{v} \in \mathbb{R}^3$. Such a set is called a *line*: intersection of two planes is a line.

Note that a system of two equations could also have no solutions or a two-dimensional space of solutions. Geometrically it means that the two planes are either parallel (and thus do not intersect) or coincide (and thus the intersection is a the plane itself).

We can also solve a converse problem: given a plane described by $\mathbf{x} = \mathbf{a} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$, write an equation of that plane. See Problem 3 below.

Homework

1. Find a polynomial $p(x)$ of degree 3 which satisfies the following conditions

$$p(0) = 5, \quad p(1) = 2, \quad p(-1) = 4, \quad p(3) = -40.$$

2. Write the equation of a plane in $\mathbb{R}^3$ passing through the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. [Hint: if the equation of the plane is $ax_1 + bx_2 + cx_3 = d$, then plugging in it each of the points gives a condition on a, b, c, d; this gives a system of linear equations.]

Is such a plane unique?

3. Let P be the plane in $\mathbb{R}^3$ described parametrically, as the set of all points of the form

$$\mathbf{x} = t_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad t_1, t_2 \in \mathbb{R}.$$

Write an equation of this plane. [Hint: find 3 points on this plane.]

4. Consider two planes $2x + 3yz = 0$, $4x + z = 4$. Prove that their intersection is a line; write the line in the parametric form, by writing a generic point in the line as $\mathbf{x} = \mathbf{a} + t\mathbf{v}$ for some vector $\mathbf{v}$.

5. Find the intersection of 3 planes

$$\begin{aligned} x + 2y + 3z &= 3 \\ 3x + y + 2z &= 3 \\ 2x + 3y + z &= 3 \end{aligned}$$