MATH 10
ASSIGNMENT 12: SYSTEMS OF LINEAR EQUATIONS

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$\mathbb{R}^n$

We use notation $\mathbb{R}^n$ for the set of all points in n-dimensional space. Such a point is described by an n-tuple of numbers (coordinates) $x_1, x_2, \dots, x_n$. We will write them as a column of numbers:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We will make no distinction between a point and a vector (starting at the origin and ending at this point). Thus, we will also refer to points of $\mathbb{R}^n$ as vectors.

We have two natural operations on $\mathbb{R}^n$: addition of vectors and multiplication by numbers:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}, \qquad c \in \mathbb{R}$$

These operations satisfy obvious associativity, commutativity, and distributivity properties.

Note that there is no multiplication of vectors — only multiplication of a vector by a number.

Systems of linear equations

We will be considering systems of linear equations such as

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 2 \\ 4x_1 - 7x_2 + 5x_3 &= 1 \end{aligned}$$

Such a system of equations is determined by the collection of coefficients, which naturally form a rectangular array (matrix), and the numbers in the right-hand side of the equation. In the example above the matrix is

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -7 & 5 \end{bmatrix}$$

and the right-hand side is $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Symbolically, we will write a system of linear equations as $A\mathbf{x} = \mathbf{b}$. We can put the matrix and the right-hand side together, forming what is sometimes called the *augmented* matrix:

$$A|\mathbf{b} = \begin{bmatrix} 2 & 1 & 3 & | & 2 \\ 4 & -7 & 5 & | & 1 \end{bmatrix}$$

Elementary row operations

The main idea of solving an arbitrary system of linear equations is by transforming it to a simpler form. Transformation should not change the set of solutions. To do this, we will use the following elementary operations:

- Exchange two equations (= two rows of the augmented matrix)
- Multiply both sides of an equation (= one row of augmented matrix) by a non-zero number
- Add to one equation a multiple of another.

Applying these operations to bring your matrix to a simpler form is called *row reduction*, or *Gaussian elimination*

Simple example

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 5 \\ x_1 - x_2 &= -1 \\ -x_1 + x_2 + x_3 &= 5 \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 1 & -2 & 2 & | & 5 \\ 1 & -1 & 0 & | & -1 \\ -1 & 1 & 1 & | & 5 \end{bmatrix}$$

Using row operations, we can bring it to the form

$$\begin{bmatrix} 1 & -2 & 2 & | & 5 \\ 0 & 1 & -2 & | & -6 \\ 0 & 0 & 1 & | & 4 \end{bmatrix}$$

so the solution is

$$\begin{aligned} x_3 &= 4 \\ x_2 &= -6 + 2x_3 = 2 \\ x_1 &= 5 - 2x_3 + 2x_2 = 1 \end{aligned}$$

Row echelon form

In general, using row operations, every system can be brought to a form where each row begins with some number of zeroes, and each next row has more zeroes than the previous one:

$$\begin{bmatrix} X & * & * & * & * & * & * & | * \\ 0 & 0 & 0 & X & * & * & * & | * \\ 0 & 0 & 0 & 0 & X & * & * & | * \end{bmatrix}$$

(here X's stand for non-zero entries).

To solve such a system, we do the following:

- Variables corresponding to columns with X's in them are called pivot variables; the remaining ones are called free variables.
- Values for free variables can be chosen arbitrarily. Values for pivot variables are then uniquely determined from the equations.

For example, in the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_2 + 3x_3 &= 6 \end{aligned}$$

varaibles x_1, x_2 are pivot, and variable x_3 is free, so we can solve it by letting $x_3 = t$, and then

$$\begin{aligned} x_2 &= 6 - 3x_3 = 6 - 3t \\ x_1 &= 5 - x_2 - x_3 = -1 + 2t \end{aligned}$$

Homework

1. Solve the following system of equations

$$\begin{aligned} w + x + y + z &= 6 \\ w + y + z &= 4 \\ w + y &= 2 \end{aligned}$$

2. Solve the system of equations with the following matrix

$$\begin{bmatrix} 2 & -1 & 3 & | & 2 \\ 1 & 4 & 0 & | & -1 \\ 2 & 6 & -1 & | & 5 \end{bmatrix}$$

3. Solve the following system of equations

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 3 \\ -x_1 + x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + 8x_3 &= 4 \end{aligned}$$

4. Consider the system of equations

$$\begin{aligned} 3x - y + 2z &= b_1 \\ 2x + y + z &= b_2 \\ x - 7y + 2z &= b_3 \end{aligned}$$

(a) If $b_1 = b_2 = b_3 = 0$, find all solutions
(b) For which triples b_1, b_2, b_3 does it have a solution?

5. Consider a system of 4 equations in 5 variables.
(a) Show that if the right-hand side is zero, then this system must have a non-zero solution.
(b) Is it true if the right-hand side is non-zero?

6. If we have a system of k equations in n variables, how many free variables will there be? How many parameters will there be for a general solution?