

**MATH 10**  
**ASSIGNMENT 10: CONTINUOUS FUNCTIONS**  
JAN 7, 2018

**Definition.** A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is called continuous if, for every sequence  $a_n \in \mathbb{R}$  which has a limit:  $\lim a_n = A \in \mathbb{R}$ , the sequence  $f(a_n)$  also has a limit and  $\lim f(a_n) = f(A)$ .

For example, function  $f(x) = x$  is continuous, while the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

is not (see Problem 1 below)

Instead of functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ , the same definition can be applied to functions between other sets: if  $X, Y$  are metric spaces (i.e., have sets with the notion of distance, satisfying all required properties), then the above definition works without any changes for functions  $f: X \rightarrow Y$ . For example, we can talk about continuous functions on an interval  $[0, 1]$  or on the set  $\mathbb{R}_+$  of positive real numbers. Note that in the latter case we only require that  $f(a_n)$  converge for a sequence  $a_n \in \mathbb{R}_+$  which has a limit also in  $\mathbb{R}_+$ . For example, we do not require that  $f(1/n)$  converge.

HOMEWORK

1. Prove that the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

is not continuous.

2. Prove that the function  $f(x) = x^2 + 1$  is continuous.

3. Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions.

(a) Prove that  $f + g, f - g, fg$  are also continuous. [Hint: remember the limit laws?]

(b) Prove that  $f/g$  is continuous on the set  $X = \{x \in \mathbb{R} \mid g(x) \neq 0\} \subset \mathbb{R}$ .

(c) Deduce that any polynomial is continuous everywhere on  $\mathbb{R}$ , and a rational function  $f(x) = p(x)/q(x)$  is continuous everywhere it is defined.

4. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Prove that then the set  $A = \{x \in \mathbb{R} \mid f(x) > 0\}$  is open in  $\mathbb{R}$  (if you have forgotten the definition of open set, review Assignment 2 from September 2017). Hint: otherwise, set  $A$  contains a point  $x \in \partial A$ ; then choose a sequence  $x_n \in A'$  such that  $\lim x_n = x$ , where  $A' = \{x \in \mathbb{R} \mid f(x) \leq 0\}$  is the complement of  $A$ .

- \*5. Modify the previous proof to show that if  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function, and  $U \subset \mathbb{R}$  is open, then  $f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$  is also open.