

MATH 10
ASSIGNMENT 7: SERIES
NOVEMBER 12, 2017

SERIES

Given a sequence a_n , consider a new sequence

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\dots \\ S_n &= a_1 + \dots + a_n = \sum_{i=1}^n a_i \end{aligned}$$

If the sequence S_1, \dots, S_n has a limit, we will write

$$\sum_{i=1}^{\infty} a_i = \lim S_n$$

and call it the sum of the infinite series. In such a situation we say that the infinite series $\sum_1^{\infty} a_n$ converges. For example:

$$1 + r + r^2 + \dots = \sum_{i=0}^{\infty} r^i = \frac{1}{1-r}, \quad |r| < 1$$

Note that it is quite possible that the sequence a_n converges but the series $\sum_1^{\infty} a_n$ does not converge!

1. Prove that if the series $\sum_1^{\infty} a_n$ converges, i.e. the limit $\lim S_n$ exists, then $\lim a_n = 0$. [Hint: $a_n = S_n - S_{n-1}$.]
2. Prove that if $0 \leq a_n \leq b_n$, then
 - (a) $\sum_{i=1}^n a_i \leq \sum_{i=1}^n b_i$
 - (b) If the series $\sum_{i=1}^{\infty} b_i$ converges: $\sum_{i=1}^{\infty} b_i = B$, then the series $\sum_{i=1}^{\infty} a_i$ also converges, and $\sum_{i=1}^{\infty} a_i \leq B$. [Hint: show that $S_n = \sum_{i=1}^n a_i$ is a bounded increasing sequence.]
3.
 - (a) Prove that the series $\sum \frac{1}{n(n+1)}$ converges and find the sum.
 - (b) Use the previous problem to prove that the series $\sum \frac{1}{n^2}$ converges. [This problem is essentially a repetition of the last problem in the previous HW.]

4. Prove that the harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

does not converge. Hint: group the terms as follows:

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) \dots$$

and show that the sum of terms inside each parentheses is $\geq 1/2$.

5. Prove that the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

converges, by noticing that $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$.

The value of this series is denoted by letter e and is at least as important in math as the number π :

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718281828 \dots$$

(where we use the convention $0! = 1$)