

**MATH 10**  
**ASSIGNMENT 4: LIMITS**  
OCTOBER 15, 2017

LIMITS

We say that a sequence  $a_n$  has limit  $A$  if, as  $n$  increases, terms of the sequence get closer and closer to  $A$ .

This definition is not very precise. For example, the terms of sequence  $a_n = 1/n$  get closer and closer to 0, so one expects that the limit is 0. On the other hand, it is also true that they get closer and closer to  $-1$ . So the words “closer and closer” is not a good way to express what we mean.

A better way to say this is as follows.

**Definition.** A set  $U$  is called a *trap* for the sequence  $a_n$  if, starting with some index  $N$ , all terms of the sequence are in this set:

$$\exists N : \forall n \geq N : a_n \in U$$

Note that it is not the same as “infinitely many terms of the sequence are in this set”.

Now we can give a rigorous definition of a limit.

**Definition.** A number  $A$  is called the *limit* of sequence  $a_n$  (notation:  $A = \lim a_n$ ) if for any  $\varepsilon > 0$ , the neighborhood  $B_\varepsilon(A) = \{x \mid d(x, A) < \varepsilon\}$  is a trap for the sequence  $a_n$ .

For example, when we say that for a sequence  $a_n \in \mathbb{R}$ ,  $\lim a_n = 3$ , it means:

there is an index  $N$  such that for all  $n \geq N$  we will have  $a_n \in (2.99, 3.01)$ ,

there is an index  $N'$  (possibly different) such that for all  $n \geq N'$  we will have  $a_n \in (2.999, 3.001)$

there is an index  $N''$  such that for all  $n \geq N''$  we will have  $a_n \in (3 - 0.0000001, 3 + 0.0000001)$

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1. Consider the sequence  $a_n = 1/n$ .
  - (a) Fill in the blanks in each of the statements below:
    - For all  $n \geq \underline{\hspace{1cm}}$ ,  $|a_n| < 0.1$
    - For all  $n \geq \underline{\hspace{1cm}}$ ,  $|a_n| < 0.001$
    - For all  $n \geq \underline{\hspace{1cm}}$ ,  $|a_n| < 0.00017$
  - (b) Show that  $\lim a_n = 0$ .
2. Prove that  $\lim \frac{1}{n(n+1)} = 0$  (hint:  $\frac{1}{n(n+1)} < \frac{1}{n}$ ).
3. Find the limits of the following sequences if they exist:
  - (a)  $a_n = \frac{1}{n^2}$
  - (b)  $a_n = \frac{1}{2^n}$
  - (c)  $a_n = n$
4. Explain why the number 1 is NOT a limit of the sequence  $(-1)^n$ .
5. Show that the limit of a sequence (if exists) is an accumulation point.
6. Show that the set of accumulation points of a sequence is closed.