Accelerated Math. Class work 10.

Algebra.

Power of a power.

Let us explore the definition of the exponent, a^m, when m is a positive integer. Then

$$a^n = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times} \tag{1}$$

Based on this definition (1) we can show that

$$a^n \cdot a^m = \underbrace{a \cdot a \ldots \cdot a}_{n\ times} \cdot \underbrace{a \cdot a \ldots \cdot a}_{m\ times} = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n+m\ times} = a^{n+m} \tag{2}$$

and

$$(a^n)^m = \underbrace{a^n \cdot a^n \cdot \ldots \cdot a^n}_{m\ times} = \underbrace{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times} \cdot \ldots \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}_{m\ times} = a^{n \cdot m} \tag{3}$$

If we want to multiply $a^n = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times}$ by another a we will get the following expression:

$$a^n \cdot a = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times} \cdot a = \underbrace{a \cdot a \cdot a \cdot a \ldots \cdot a}_{n+1\ times} = a^{n+1} = a^n \cdot a^1 \tag{4}$$

In order to have the set of power properties consistent, $a^1 = a$ for any number a.

We can multiply any number by 1, this operation will not change the number, so if

$$a^n = a^n \cdot 1 = a^{n+0} = a^n \cdot a^0 \tag{5}$$

In order to have the set of power properties consistent, $a^0 = 1$ for any number a.

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}}, n > m$$

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n-m\ times} = a^{n-m} \tag{6}$$

We can rewrite the expression (6) as

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = \left(\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}\right) : \left(\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}\right) = a^n : a^m = a^{n-m} \qquad (7)$$

We see that based on the definition of the exponentiation n-th and m-th powers add up if a^n and a^m are multiplied and subtract if a^n is divided by a^m. We know, on the other hand, that division is a multiplication by inverse number.

Let rewrite (6) again

$$\frac{a^n}{a^m} = a^n : a^m = a^{n+(-m)} = a^n \cdot \frac{1}{a^m} = a^n \cdot a^{-m} \ , a \neq 0 \ \text{ for any } n \text{ and whole } m \qquad (7)$$

Negative power of a number, not equal to 0, can be defined as

$$a^{-m} = \frac{1}{a^m} \ , \ a \neq 0 \qquad (8)$$

So, all properties of the exponentiation can be defined as:

1. $a^n = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times}$
2. $a^n \cdot a^m = a^{n+m}$
3. $(a^n)^m = a^{n \cdot m}$
4. $a^1 = a,$ for any a
5. $a^0 = 1,$ for any a
6. $\frac{1}{a^m} = a^{-m},$ for any $a \neq 0$

Also, if there are two numbers a and b:

$$(a \cdot b)^n = \underbrace{(a \cdot b) \cdot \ldots \cdot (a \cdot b)}_{n\ times} = \underbrace{a \cdot \ldots \cdot a}_{n\ times} \cdot \underbrace{b \cdot \ldots \cdot b}_{n\ times} = a^n \cdot b^n \qquad (9)$$

Last property is

7. $(a \cdot b)^n = a^n \cdot b^n$

Exercises

1. Evaluate the following expressions
   $9a^{2,}$, $(9a)^2$, $-9a^2$, $(-9a)^2$
   for $a = \frac{1}{6}, -\frac{1}{6}, -0.1$

2. Write the following numbers as the power of base 10:
   a. 10, 100, 1000, 10000, 100000, 1000000
   b. 0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001

3. Write in the in ascending order
   a. $-1.2, -1.2^2, 1.2, (-1.2)^2$
   b. $0.15, -0.15, (-0.15)^2, (-0.15)^3$

   We saw in the previous examples, that the sign of the results of exponentiation with the power represented by a natural number can be easily defined. Let's prove a few statements.

I. A positive number raised into any power will result a positive number.
   - Because the product of any number of positive factors gives as outcome a positive number, and exponentiation is a product of same factors, we proved the statement.

II. A negative number, raised in a power, represented by an even number is positive, represented by an odd number is negative.
   - Even number m of negative factors produces $\frac{m}{2}$ pairs of multiplied negative numbers each of which is positive, as we know, of product any quantity of positive factors is positive.
   - For an odd number n, $\frac{n-1}{2}$ pairs of negative factors will be multiply by one more negative number, so the whole product will be negative.

Exercises

4. Simplify the following expressions:
   a. $aa^m(-a)^2$;
   b. $c^k c(-c^2)c^{k-1}c^3$;
   c. $d^n d(-d^{n+1})d^n d^2$;
   d. $2x^2y^3 \cdot (-4xy^2)$;
   e. $0.5a(-b)^6 \cdot 10a^2b^2$;
   f. $\frac{1}{6}(-c)^3dk \cdot (-6cdk^3)$;
   g. $2^4 + 2^4$;
   h. $2^m + 2^m$;
   i. $2^m \cdot 2^m$;
   j. $3^2 + 3^2 + 3^2$;
   k. $3^k + 3^k + 3^k$;
   l. $3^k \cdot 3^k \cdot 3^k$;

5. Simplify:

$$\frac{3xy\cdot\frac{2}{5}xz-2x\cdot xyz-\frac{1}{3}x^2yz+x-(5+2x-7)+x-2}{2xy-2yz\cdot z-xy+2yz\cdot y+\frac{13}{15}z^2y-4zy^2+2zy\cdot y-xy};$$

Geometry.

In geometry, two figures or objects are **congruent** if they have the same shape and size, or if one has the same shape and size as the mirror image of the other.[1]

More formally, two sets of points are called **congruent** if, and only if, one can be transformed into the other by a combination of **rigid motions**, namely a translation, a rotation, and a reflection. This means that either object can be repositioned and reflected (but not resized) so as to coincide precisely with the other object. So, two distinct plane figures on a piece of paper are congruent if we can cut them out and then match them up completely. Turning the paper over is permitted.

In elementary geometry the word *congruent* is often used as follows. The word *equal* is often used in place of *congruent* for these objects.

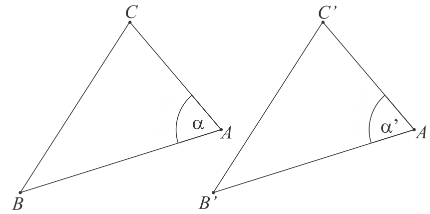
- Two line segments are congruent if they have the same length.
- Two angles are congruent if they have the same measure.
- Two circles are congruent if they have the same diameter.

Sufficient evidence for congruence between two triangles can be shown through the following comparisons:

- **SAS** (Side-Angle-Side): If two pairs of sides of two triangles are equal in length, and the included angles are equal in measurement, then the triangles are congruent.

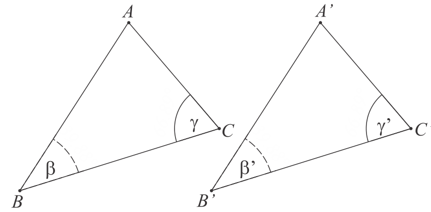
- **SSS** (Side-Side-Side): If three pairs of sides of two triangles are equal in length, then the triangles are congruent.

- **ASA** (Angle-Side-Angle): If two pairs of angles of two triangles are equal in measurement, and the included sides are equal in length, then the triangles are congruent.



SAS (Side-Angle-Side).
ABC and A'B'C' are two triangles such that

$AC = A'C', AB = A'B'$, and $\angle A = \angle A'$ We need to prove that these triangles are congruent. Superimpose ΔABC onto $\Delta A'B'C'$ in such a way that vertex A would coincide with A', the side AC would go along $A'C'$, and side AB would lie on the same side of $A'C'$ as $A'B'$. Since AC is congruent to $A'C'$, the point C will merge with point C'., due to the congruence of $\angle A$ and $\angle A'$, the side AB will go along $A'B'$ and due to the congruence of these sides, the point B will coincide with B'. Therefor the side BC will coincide with $B'C'$.

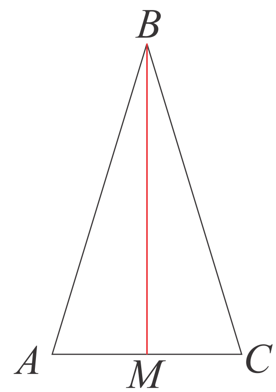


ASA (Angle-Side-Angle) ABC and $A'B'C'$ are two triangles, such that

$\angle C = \angle C', \angle B = \angle B'$, and $BC = B'B'$.

We need to prove, that these triangles are congruent. Superimpose the triangles in such a way that point C will coincide with point C', the side CB would go along $C'B'$ and the vertex A would lie on the same side of $C'B'$ as A'. Then: since CB is congruent to $C'B'$, the point B will merge with B', and due to congruence of the angle $\angle B$ an $\angle B'$, and $\angle C$ and $\angle C'$, the side BA will go along B'A', and side CA will go along C'A'. Since two lines can intersect only at 1 point, the vertex A will have merge with A'. Thus, the triangles are identified and are congruent.

Theorem. In isosceles triangle the bisector passed to the base (in isosceles triangle the base is the side different from two equal sides) is a median and an altitude as well.



Let the triangle ΔABC be an isosceles triangle, such that $AB = BC$, and BM is a bisector. We need to prove that BM is a median and an altitude, which means that $AM = MC$ and angle $\angle BMC$ is a right angle.

BM is a bisector, so $\angle ABM$ and $\angle MBC$, the triangle ΔABM is an isosceles triangle, so $AB = BC$ and the segment MB is common side for triangles ΔABM and ΔMBC. Based on the Side-Angle-Side criteria, the triangles ΔABM and ΔMBC are congruent. Therefore, $AM = MC$ (BM is a median), angles $\angle A$ and $\angle C$ are congruent. (Isosceles triangle has equal angles adjacent to the base).

$\angle A + \angle B + \angle C = 180° = 2\angle A + \angle B \Rightarrow 90° = \angle A + \frac{1}{2}\angle B$ but for the triangle ABM (as well as for MBC), $\angle A + \frac{1}{2}\angle B + \angle BMA = 180°$, therefore $\angle BMA = 90°$and BM is also an altitude.

Exercise.

On one side of an angle $\angle A$, the segments AB and AC are marked, and on the other side the segments $AB' = AB$ and $AC' = AC$. Prove that the lines BC' and $B'C$ met on the bisector of the angle $\angle A$.