Homework 15

Lorentz transformations.

The result of Michelson-Morley experiment became clear after the work published by A. Einstein in 1905. In this work he demonstrated the constancy of the speed of light in all inertial reference frames.

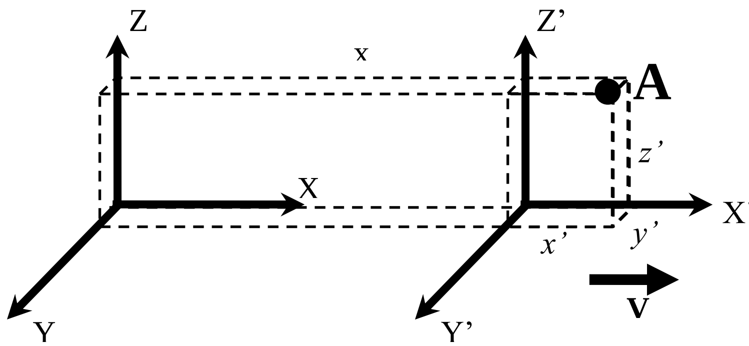

Figure 1

Let us consider a coordinate system $X'Y'Z'$ which is moving at a velocity $\boldsymbol{V}$ with respect to the XYZ frame along the x axis (Fig.1). Let us imagine that the object A has coordinates x,y,z in the XYZ reference frame. We are going to express the coordinates x', y', z' of A in $X'Y'Z'$ reference frame. According to classical nonrelativistic mechanics the transformation rules are:

$$
\begin{aligned}
x' &= x - Vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
\qquad (1)
$$

The last equation looks absolutely trivial and intuitive. It means that the time flow is the same in both reference frames. However, the Michelson-Morley experiment suggested that the speed of light is the same in all inertial reference frames. This is in contradiction with the above coordinate transformations. Einstein demonstrated that the correct formulae are:

$$
\begin{aligned}
x' &= \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} \\
y' &= y \\
z' &= z \\
t' &= \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}
\end{aligned}
\qquad (2)
$$

Here, c is the light speed in vacuum. These formulae – Lorentz transformations - were originally suggested by Hendrick Lorentz (1899) and, independently, by Joseph Larmor (1897).

Albert Einstein
(1879-1955).

There are 3 very interesting and counterintuitive conclusions which we can come to analyzing Lorentz transformations:

1. Length contraction

Let us imagine a rod lying at rest along the x' axis of the reference frame $X'Y'Z'$, moving at a constant velocity *V*. The coordinate of the ends of the rod in this reference frame are x'_1 and x'_2, so the length of the rod in this reference frame is $\Delta x' = x'_2 - x'_1$ .Using the Lorentz transformation (2) we can find the how the coordinates of the rod ends in a moving reference frame are related to these in the frame at rest:

$$\Delta x' = \frac{\Delta x - V\Delta t}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (3)$$

What is the rod length in the frame at rest? It is natural to assume that it is $\Delta x = x_2 - x_1$, nut both x_2 and x_1 are *measured at the same moment of time*, so we can set Δt in the formula (3) to be equal to zero:

$$\Delta x' = \frac{\Delta x}{\sqrt{1-\frac{V^2}{c^2}}}, or\ \Delta x = \Delta x' \cdot \sqrt{1-\frac{V^2}{c^2}} \qquad (4).$$

This means that all the objects which are moving with respect to us *contract in the direction of motion*. But this contraction is relative: the person in the moving reference frame will see that all objects in the frame at rest contract. The dimensions in the directions perpendicular to the velocity are unaffected.

2. Time dilation

Another amazing conclusion which follows from analysis of the Lorentz transforms is that we cannot "agree on time" anymore. There is no absolute time as it was in Galilean mechanics. Now the time "flow" depends on the velocity of the reference of frame.

Let us consider a clock C at position x'_0 in a moving reference frame $X'Y'Z'$ (Figure 2). This clock measures time interval $\Delta t' = t'_2 - t'_1$ for those who travel together with clock C. It may be a time interval separating any two events happening in frame $X'Y'Z'$at the same place. We will use as the events the arriving of the clock handles to a certain positions. Now, how does this time interval is measured in the frame *XYZ,* which is at rest? To do such measurements we need a bunch of equally spaced different synchronized clocks in frame XYZ (Figure2). We see, that at the moment of the first event, the clock C is flying past clock 1 in the rest frame *XYZ* . The reading of clock 1 we take as t_1. At the moment of the second event (as we see from our rest frame) clock C is flying by clock 2. The reading of clock 1 we take as t_2. Then we find $\Delta t = t_2 - t_1$. Time interval Δt in the rest frame is related to time interval $\Delta t'$ in the moving frame as:

$$\Delta t = \frac{\Delta t' + \frac{V}{c^2}\Delta x'}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (3)$$

Please note that "minus" sign is changed to "plus" since for any person in a moving frame, the rest frame is moving at a velocity -*V*.

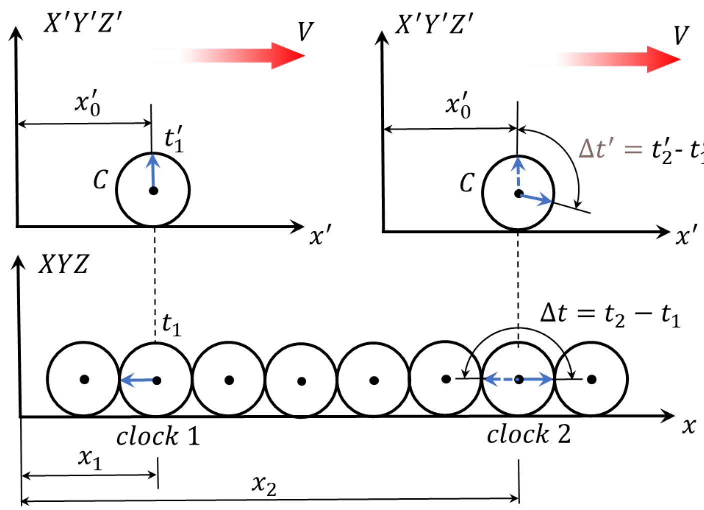

Figure 2.

As long as in the moving frame clock C is at rest, we have to set $\Delta x' = 0$:

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{V^2}{c^2}}}$$

So, from our perspective, the moving clock slow down. If your, say, classmate is moving with respect to you, then from your point of view his or her clock runs slower. But your classmate is confident in that it is *your* clock which is slow. For him or her time is going normally. So even if you are traveling in space at a very high speed and, for those who are left on the Earth your time is much slower than the Earth time, you will not enjoy extended life: in your reference frame time is "normal" as well as aging, unfortunately.

Problems:

In the problems below ***c*** is the speed of light in vacuum.

1. How do Lorentz transformation (equations (2)) look from the point of view of a person in the moving reference frame $X'Y'Z'$?

2. Imagine that an astronaut is moving at a speed 0.8c relative to the Earth. Find how long is his (her) hour as it is "seen" from the Earth?

3. Two events happen at the same time in a certain reference frame. Do you think that these events are simultaneous in all inertial reference frames? Proof your answer.

4. In a certain reference frame two events happen at different moments of time in different places. Is it possible to find another reference frame in which these events will happen at the same place?