Algebra.

Complex numbers.

Let us consider the space $\mathbb{R}[i]$ of polynomials in one variable (which we will now denote by i rather than x) but with one extra relation:

$i^2 + 1 = 0$

Thus, we will treat two polynomials in i, which differ by a multiple of $i^2 + 1$ as equal. This can be done more formally in the same way as multiplication and division of remainders modulo n for integers. Note that this relation implies,

$i^2 = -1,\ i^3 = i^2 i = -i,\ i^4 = 1,\ \dots,\ i^{2n} = (-1)^n,\ i^{2n+1} = (-1)^n i$.

So, using this relation we can replace any polynomial in i by a linear polynomial of the form $a + bi$. For example,

$P_5(i) = a_5 i^5 + a_4 i^4 + a_3 i^3 + a_2 i^2 + a_1 i + a_0 = a_4 - a_2 + a_0 + (-a_3 + a_1)i$, or,

$(1 - i)(2 + 3i) = 5 + i$, etc. Also, $i^n = i^{n\, mod 4}$.

Definition. The set $\mathbb{C}$ of complex numbers is the set of "numbers" of the form $a + bi,\ a, b \in \mathbb{R}$, with addition and multiplication same as for usual polynomials, and with added relation $i^2 = -1$. The "number" i is often called imaginary unit.

Note that any real number a can also be considered as a complex number by writing it as $a + 0i$; thus, $\mathbb{R} \subset \mathbb{C}$. Consequently, for any complex number, $z = a + bi$, the number a is called real part of z and denoted $a = \mathrm{Re}\, z$; the number b is called imaginary part of z and denoted $b = \mathrm{Im}\, z$. Complex numbers of the form $z = bi$ whose real part is zero are called "imaginary numbers". This name was coined in the 17th century as a derogatory term, as such numbers were regarded by some as fictitious or useless.

Since multiplication and addition of polynomials satisfies the usual distributive and commutative properties, the same holds for complex numbers.

In addition to multiplication, we can further define the inverse of a complex number, and therefore the division of complex numbers.

Definition. A conjugate of a complex number, $z = a + bi$, is $\bar{z} = a - bi$.

Theorem. The operation of conjugation of a complex number, $z \to \bar{z}$ has the following properties:

1. $\overline{(z + w)} = \bar{z} + \bar{w}$
2. $\overline{(zw)} = \bar{z}\bar{w}$

For any complex number, $z = a + bi$, its product with its conjugate, $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$, is a non-negative real number.

Definition. An absolute value of a complex number, $z = a + bi$, is a non-negative real number $|z|$, such that $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2} \Leftrightarrow |z|^2 = z\bar{z}$.

Using the conjugation properties, it is easy to see that for any two complex numbers, z, w, $|zw| = |z||w|$.

Definition. The inverse of a non-zero complex number, $z = a + bi$, is a complex number, z^{-1}, such that $zz^{-1} = 1$.

Theorem. For any non-zero complex number, $z = a + bi$, there exists an inverse, z^{-1}, and $z^{-1} = \frac{\bar{z}}{|z|^2}$.

Exercise 1. Compute:

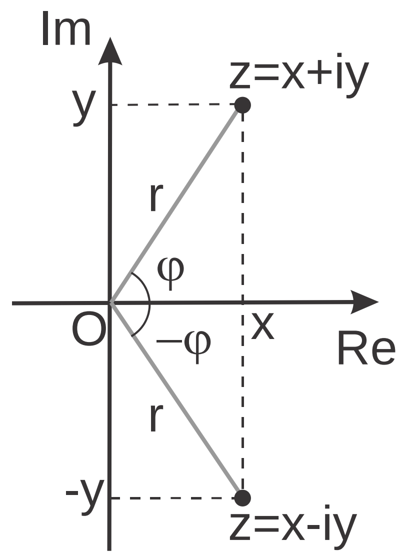
a. $(1 + i)^{-1}$
b. $\frac{1+i}{1-i}$
c. $\frac{1}{4+3i}$
d. $(1 + i)^{-3}$

Exercise 2. Solve the following equations in complex numbers:

a. $z^2 = i$
b. $z^2 = -2 + 2i\sqrt{3}$
c. $z^2 + z + 1 = 0$
d. $z^2 - z + 1 = 0$

Exercise 3. Prove that if a complex number, z, is a root of a quadratic polynomial with real coefficients, $P_2(z) = 0$, then its complex conjugate, $\bar{z}$, is also a root of this polynomial, $P_2(\bar{z}) = 0$. Prove this also for the cubic polynomial, $P_3(z)$.

Geometrical interpretation of complex numbers.



There is an obvious one-to-one correspondence between the elements, $z = x + iy$, of a set of all complex numbers $\mathbb{C}$, and the points on the coordinate plane, (x, y), see Figure. In the context of complex numbers, the coordinate plane is called the complex plane. The horizontal X-axis corresponds to the set of real numbers, $x \in \mathbb{R}$, while the vertical Y-axis corresponds to the set of purely imaginary numbers, $iy, y \in \mathbb{R}$.

In this representation, the absolute value of a complex number, $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$, is just the distance of the corresponding point, $Z(x, y)$, from the origin, O, i.e the length of the vector $\overrightarrow{OZ}$. The following properties of the operations with complex numbers immediately follow:

- Addition of two complex numbers, $Z_1(x_1, y_1), z_1 = x_1 + iy_1$, and , $Z_2(x_2, y_2), z_2 = x_2 + iy_2$, corresponds to the addition of the respective vectors: $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2), \overrightarrow{OZ_1} + \overrightarrow{OZ_2} = (x_1 + x_2, y_1 + y_2)$.
- Subtraction of two complex numbers, $Z_1(x_1, y_1), z_1 = x_1 + iy_1$, and , $Z_2(x_2, y_2), z_2 = x_2 + iy_2$, corresponds to the subtraction of the respective vectors: $z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2), \overrightarrow{OZ_1} - \overrightarrow{OZ_2} = (x_1 - x_2, y_1 - y_2)$.
- Complex conjugation, $z \to \bar{z}$, corresponds to the reflection about X-axis.
- Multiplication of a complex number by a non-negative real number, k, results in a vector, which has the same direction, but whose length is multiplied by k.
- Multiplication of a complex number by -1 changes the direction of the corresponding vector.
- Multiplication of a complex number, $Z(x, y), z = x + iy$, by i, changes it to $W(-y, x), w = -y + ix$, which corresponds to consecutive reflections with respect to the X-axis and the $y = x$ line, or, a rotation by 90°.