Homework for January 28, 2024.

## Algebra.

Review the classwork handout. Review and solve the classwork exercises which were not solved (some are repeated below). Solve the following problems (skip the ones you already solved).

1. Present examples of binary relations that are, and that are not equivalence relations.
2. For each of the following relations, check whether it is an equivalence relation and describe all equivalence classes.
   a. On $\mathbb{R}$: relation given by $x \sim y$ if $|x|=|y|$
   b. On $\mathbb{Z}$: relation given by $a \sim b$ if $a \equiv b \bmod 5$
   c. On $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$; describe the equivalence class of $(1, 2)$
   d. Let $\sim$ be the relation on the set of all directed segments in the plane defined by $\overrightarrow{AB} \sim \overrightarrow{A'B'}$ if $ABB'A'$ is a parallelogram.
   e. On the set of pairs of integers, $\{(a, b), a, b \in \mathbb{Z}, b \neq 0\}$, $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 b_2 = a_2 b_1$. Describe these equivalence classes. Is the set of the obtained equivalence classes countable?
3. Let $f: X \xrightarrow{f} Y$ be a function. Define a relation on $X$ by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Prove that it is an equivalence relation. Describe the equivalence classes for the equivalences defined by the following functions on $\mathbb{R}$.
   a. $f(x) = x^2$: $x \sim y$ if $x^2 = y^2$.
   b. $f(x) = \sin x$: $x \sim y$ if $\sin x = \sin y$.
4. Find the following sum. What is the smallest value of this sum for $x \in \mathbb{R}$?

$$\left(x - \frac{1}{x}\right)^2 + \left(x^2 - \frac{1}{x^2}\right)^2 + \cdots + \left(x^n - \frac{1}{x^n}\right)^2$$

5. The lengths of the sides of a triangle are three consecutive terms of the geometric series. Is the common ratio of this series, $q$, larger or smaller than 2? What is this ratio? What can you say about this triangle?
6. Solve the following equation,

$\frac{x-1}{x} + \frac{x-2}{x} + \frac{x-3}{x} + \cdots + \frac{1}{x} = 3$, where $x$ is a positive integer.

7. Find the following sum,
    a. $1 + 2 \cdot 3 + 3 \cdot 7 + \cdots + n \cdot (2^n - 1)$
    b. $1 \cdot 3 + 3 \cdot 9 + 5 \cdot 27 + \cdots + (2n - 1) \cdot 3^n$
8. What is the minimum value of the expression, $(1 + x)^{36} + (1 - x)^{36}$ in the interval $|x| \leq 1$?

## Geometry.

Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework.

## Problems.

1. Review derivation of the equation describing an ellipse and derive in a similar way,
    a. Equation of an ellipse, defined as the locus of points P for which the distance to a given point (focus $F_2$) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.
    b. Equation of a hyperbola, defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$.
2. Find (describe) set of all points formed by the centers of the circles that are tangent to a given circle of radius $r$ and a line at a distance $d > r$ from its center, $O$.
3. Using the method of coordinates, prove that the geometric locus of points from which the distances to two given points have a given ratio, $q \neq 1$, is a circle.
4. Find the equation of the locus of points equidistant from two lines, $y = ax + b$ and $y = mx + n$, where $a, b, m, n$ are real numbers.
5. Find the distance between the nearest points of the circles,
    a. $(x - 2)^2 + y^2 = 4$ and $x^2 + (y - 1)^2 = 9$
    b. $(x + 3)^2 + y^2 = 4$ and $x^2 + (y - 4)^2 = 9$
    c. $(x - 2)^2 + (y + 1)^2 = 4$ and $(x + 1)^2 + (y - 3)^2 = 5$
    d. $(x - a)^2 + y^2 = r_1^2$ and $x^2 + (y - b)^2 = r_2^2$