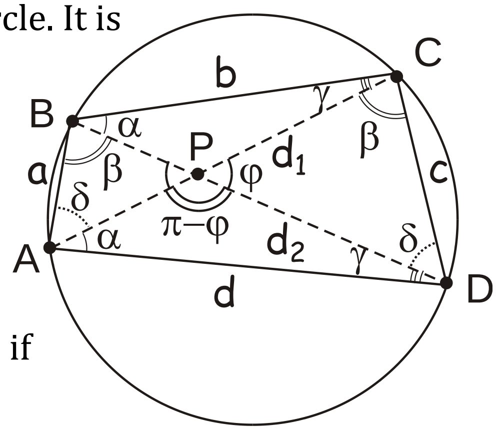
December 17, 2023

Geometry.

Properties of inscribed quadrilaterals. Ptolemy's theorem.



Consider the quadrilateral $ABCD$ inscribed into a circle. It is clear from the theorem on the inscribed angle that the opposite angles of $ABCD$ are supplementary (i. e. add to 180 degrees),

$$\hat{A} + \hat{C} = \hat{B} + \widehat{D} = \pi$$

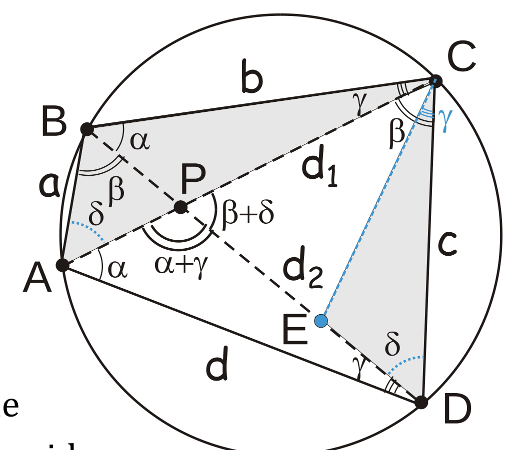
Theorem. A quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary.

Now consider angles $\alpha, \beta, \gamma, \delta$, between the sides and the diagonals. The angle between the diagonals, $\varphi = \alpha + \gamma = \pi - (\beta + \delta)$.

Theorem (Ptolemy). A quadrilateral can be inscribed in a circle if and only if the product f its diagonals equals the sum of the products of its opposite sides,

$$d_1 d_2 \ = \ ac + bd \qquad (1)$$

<u>Proof of the necessary condition of Ptolemy's theorem</u>, i.e. of Eq. (1) for an inscribed quadrilateral.



Geometrical proof employs an elegant supplementary construct. Inventing such an additional geometrical element is one of the key, most important and powerful methods of geometrical proof.

Draw segment CE, whose endpoint, E, belongs to the diagonal BD, and which is at an angle $\gamma = \widehat{ACB}$ to the side CD. Thus obtained $\Delta DEC \sim \Delta ABC$. Therefore, $\frac{|AC|}{c} = \frac{a}{|ED|}$.

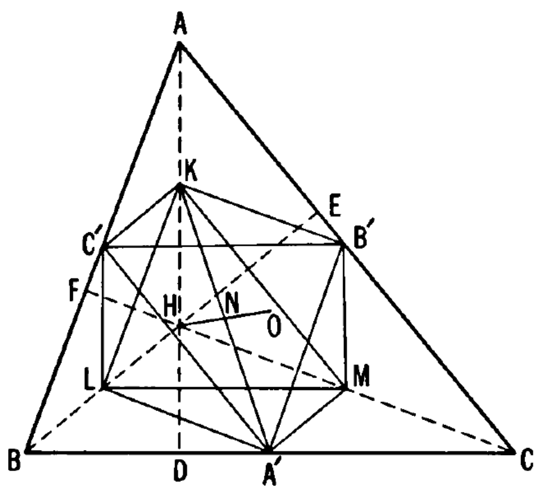
Furthermore, $\widehat{BCE} = \widehat{ACD} = \beta$ and therefore $\Delta BCE \sim \Delta ACD$, so $\frac{|AC|}{d} = \frac{b}{|BE|}$.
Adding thus obtained equalities, we get

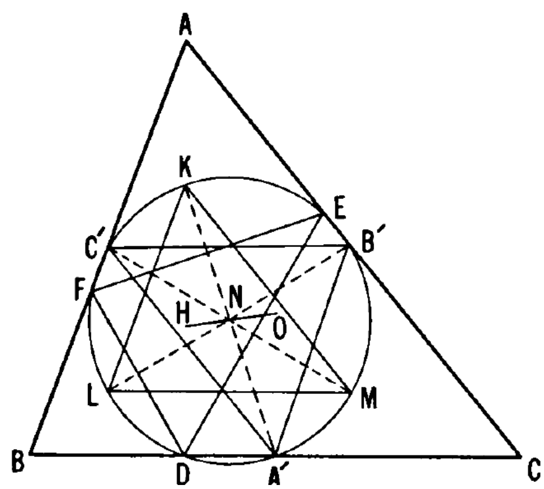
$$ac + bd = |AC||ED| + |AC||BE| = d_1 d_2.$$

The sufficiency of this condition can be easily proven by contradiction.

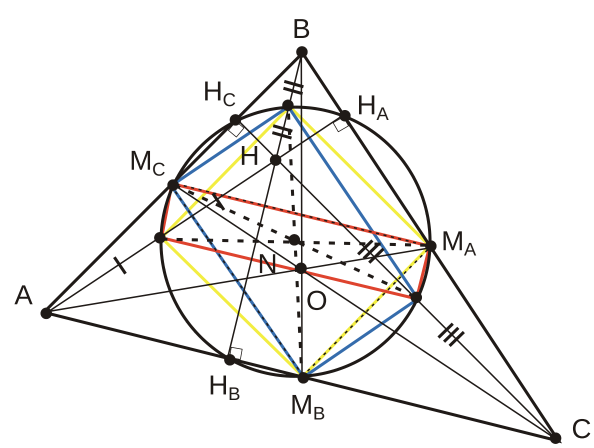
The nine-points circle problem.

Theorem. The feet of the three altitudes of any triangle, the midpoints of the three sides, and the midpoints of the segments from the three vertices lo the orthocenter, all lie on the same circle, of radius ½R.





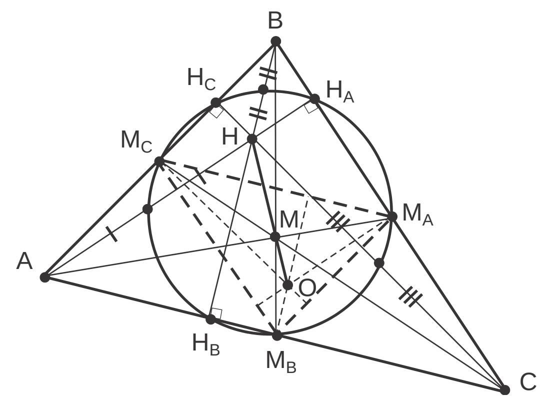
This theorem is usually credited to a German geometer Karl Wilhelm von Feuerbach, who actually rediscovered the theorem. The first complete proof appears to be that of Jean-Victor Poncelet, published in 1821, and Charles Brianson also claimed proving the same theorem prior to Feuerbach. The theorem also sometimes mistakenly attributed to Euler, who proved, as early as 1765, that the orthic triangle and the medial triangle have the same circumcircle, which is why this circle is sometimes called "the Euler circle". Feuerbach rediscovered Euler's partial result even later and added a



further property which is so remarkable that it has induced many authors to call the nine-point circle "the Feuerbach circle".

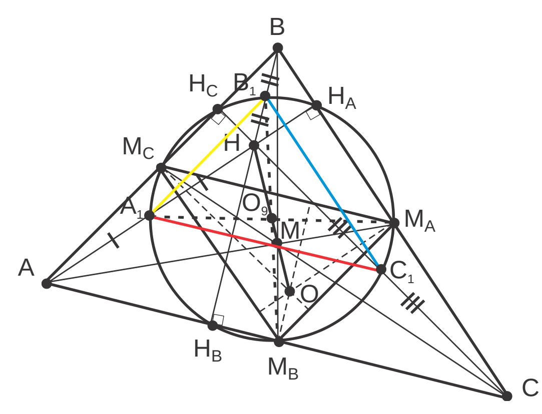
Proof. Consider rectangles formed by the mid-lines of triangle ABC and of triangles ABH, BCH and ACH.

Theorem. The orthocenter, H, centroid, M, and the circumcenter, O, of any triangle are collinear: all these three points lie on the same line, OH, which is called the Euler line of the triangle. The centroid divides the distance from the orthocenter to the circumcenter in 2: 1 ratio.



Proof. Note that the altitudes of the medial triangle$M_AM_BM_C$ are the perpendicular bisectors of the triangle ABC, so the orthocenter of $\Delta\ M_AM_BM_C$ is the circumcenter, O, of $\Delta\ ABC$. Now, using the property that centroid divides medians of a triangle in a 2:1 ratio, we note that triangles BMH and M_BMO are similar, and homothetic with respect to point M, with the homothety coefficient 2.

Theorem. The center of the nine-point-circle lies on the (Euler's) line passing through orthocenter, centroid, and circumcenter, midway between the orthocenter and the circumcenter.



Proof. Consider the figure. Note the colored triangle $A_1B_1C_1$, which is formed by medians of triangles ABH, BHC and CHA, and is therefore congruent to the medial triangle $M_AM_BM_C$, but rotated 180 degrees. The 9 points circle is the circumcircle for both triangles, which means that rotation by 180 degrees about the center O_9 of the 9 point circle moves $\Delta\ M_AM_BM_C$ onto $\Delta\ A_1B_1C_1$, and the orthocenter, O, of the $\Delta\ M_AM_BM_C$ onto the orthocenter, H, of the $\Delta\ A_1B_1C_1$.