Homework for December 10, 2023.

Algebra.

Review the classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems (some problems are repeated from previous homework – skip the ones you have already solved).

1. Rewrite the following properties of set algebra and partial ordering operations on sets in the form of logical propositions, following the first example.
   a. $[\mathrm{A} \cdot (\mathrm{B} + \mathrm{C}) = \mathrm{A} \cdot \mathrm{B} + \mathrm{A} \cdot \mathrm{C}] \Leftrightarrow \left[(\mathrm{x} \in \mathrm{A}) \wedge \big((\mathrm{x} \in \mathrm{B}) \vee (\mathrm{x} \in \mathrm{C})\big)\right] = \left[\big((\mathrm{x} \in \mathrm{A}) \wedge (\mathrm{x} \in \mathrm{B})\big) \vee \big((\mathrm{x} \in \mathrm{A}) \wedge (\mathrm{x} \in \mathrm{C})\big)\right]$
   b. $A + (B \cdot C) = (A + B) \cdot (A + C)$
   c. $(A \subset B) \Leftrightarrow A + B = B$
   d. $(A \subset B) \Leftrightarrow A \cdot B = A$
   e. $(A + B)' = A' \cdot B'$
   f. $(A \cdot B)' = A' + B'$
   g. $(A \subset B) \Leftrightarrow (B' \subset A')$
   h. $(A + B)' = A' \cdot B'$
   i. $(A' + B')' + (A' + B\,)' = A$
2. Using definitions from the classwork handout, devise logical arguments proving each of the following properties of algebra and partial ordering operations on sets and draw Venn diagrams where possible (hint: use problem #1).
   a. $\mathrm{A} \cdot (\mathrm{B} + \mathrm{C}) = \mathrm{A} \cdot \mathrm{B} + \mathrm{A} \cdot \mathrm{C}$
   b. $A + (B \cdot C) = (A + B) \cdot (A + C)$
   c. $(A \subset B) \Leftrightarrow A + B = B$
   d. $(A \subset B) \Leftrightarrow A \cdot B = A$
   e. $(A + B)' = A' \cdot B'$
   f. $(A \cdot B)' = A' + B'$
   g. $(A \subset B) \Leftrightarrow (B' \subset A')$
   h. $(A + B)' = A' \cdot B'$
   i. $(A' + B')' + (A' + B\,)' = A$
3. Verify that a set of eight numbers, {1,2,3,5,6,10,15,30}, where addition is identified with obtaining the least common multiple,

$$m + n \equiv LCM(n, m)$$

multiplication with the greatest common divisor,

$$m \cdot n \equiv GCD(n, m)$$

$m \subset n$ to mean "m is a factor of n",

$$m \subset n \equiv \left(n = 0 mod(m)\right)$$

and

$$n' \equiv 30/n$$

satisfies all laws of the set algebra.

4. For a set A, define the characteristic function χ_A as follows,
$$\chi_A(x) = \begin{cases} 1, if\ x \in A \\ 0, if\ x \notin A \end{cases}$$
Show that χ_A has following properties
$$\chi_A = 1 - \chi_{A'}$$
$$\chi_{A\cap B} = \chi_A\, \chi_B$$
$$\chi_{A\cup B} = 1 - \chi_{A'\cap B'} = 1 - \chi_{A'}\, \chi_{B'} = 1 - (1 - \chi_A)(1 - \chi_B)$$
$$= \chi_A + \chi_B - \chi_A \chi_B$$
Write formulas for $\chi_{A\cup B\cup C}$, $\chi_{A\cup B\cup C\cup D}$.

5. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.

6. Using the continued fraction representation, find rational number, r, approximating $\sqrt{2}$ to the absolute accuracy of 0.0001.

7. Consider the values of the following expression, y, for different x. How does it depend on x when n becomes larger and larger?

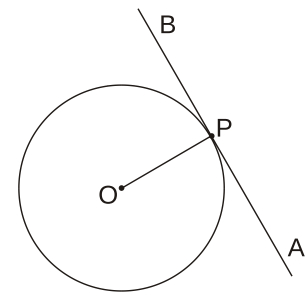
$$n\ fractions \left\{ \begin{array}{l} y = 3 - \cfrac{2}{3 - \cfrac{2}{3 - \cfrac{2}{3 - \cdots}}} \\ \qquad\qquad\qquad \cdots - \cfrac{2}{3 - x} \end{array} \right. .$$

Geometry.

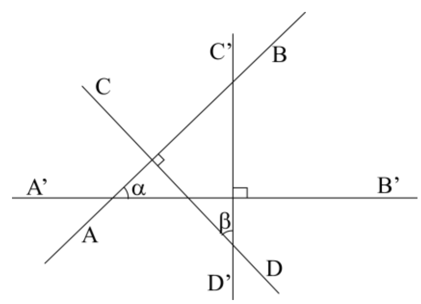
Review the last classwork handout on solving problems using mass points and the center of mass. Solve the unsolved problems from previous homeworks. Try solving the following problems (some problems are repeated from previous homework – skip the ones you have already solved).

Problems.

1. **Tangent line** to a circle is a line that has one and only one common point with the circle (definition). Prove that tangent line AB is perpendicular to the radius OP ending at the point P, which is the common point of the line and the circle (see Figure on the right).



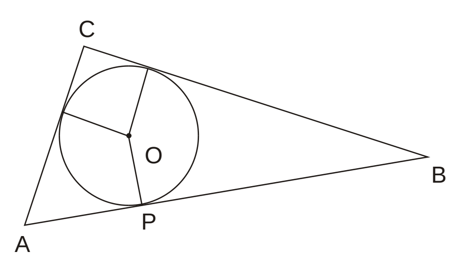
2. We know from geometry that a circle can be drawn through the three vertices of any triangle. Find a radius of such circle if the sides of the triangle are 6, 8, and 10. (Gelfand and Saul "Trigonometry" p60, #4).
3. Prove that in the Figure on the right, $\angle\alpha$ is congruent to $\angle\beta$ if $AB \perp CD$ and $A'B' \perp C'D'$.



4. Using a compass and a ruler, draw a circle inscribed in the given triangle ABC. Prove the following formula for the area of the triangle,

$$S_{ABC} = ½pr,$$

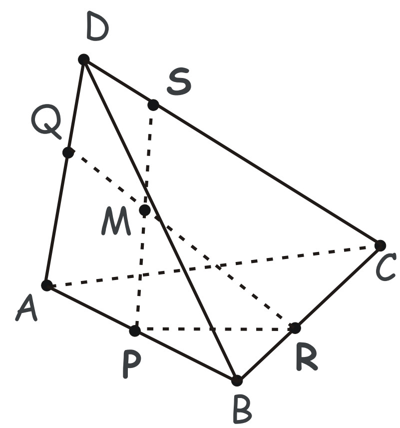
where p is the perimeter of the triangle and r the radius of the inscribed circle.



5. Prove the Viviani's theorem:

The sum of distances of a point P inside an equilateral triangle or on one of its sides, from the sides, equals the length of its altitude. Or, alternately, From a point P inside (or on a side) of an equilateral triangle ABC drop perpendiculars PP_a, PP_b, PP_c to its sides. The sum $|PP_a| + |PP_b| + |PP_c|$ is independent of P and is equal to any of the triangle's altitudes.

6. * Three Points are taken at random on an infinite plane. Find the chance of their being the vertices of an obtuse-angled Triangle. Hint: use the Viviani's theorem.
7. Consider segments connecting each vertex of the tetrahedron $ABCD$ with the centroid of the opposite face (the crossing point of its medians). Prove that all four of these segments, as well as the segments connecting the midpoints of the opposite edges (opposite edges have no common points; there are three pairs of opposite edges in a tetrahedron, and therefore three such segments) – seven segments in total, have common crossing point (are concurrent).



8. In a quadrilateral $ABCD$, E and F are the mid-points of its diagonals, while O is the point where the midlines (segments conneting the midpoints of the opposite sides) cross. Prove that E, F, and O are collinear (belong to the same line).
9. What is the ratio of the two segments into which a line passing through the vertex A and the middle of the median BB' of the triangle ABC divides the median CC'?
10. In a parallelogram $ABCD$, a line passing through vertex D passes through a point E on the side AB, such that $|AE|$ is $1/n$-th of $|AB|$, n is an integer. At what distance from A, relative to the length, $|AC|$, of the diagonal AC it meets this diagonal?
11. Points P and Q on the lateral sides AB and BC of an isosceles triangle ABC divide these sides into segments whose lengths have ratios $|AP|:|PB| = n$, and $|BQ|:|QC| = m$. Segment PQ crosses altitude BB' at point M. What is the ratio $|BM|:|MB'|$ of two segments into which PQ divides the altitude BB'?