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Algebra. Basic Notations.

Elements of Mathematical Logic.

Proposition is a sentence that is either true, or false, but not both. For example,

"Grass is green", and "$2+5=5$" are propositions.

"Close the door", and "Is it hot outside?" are not propositions.

Also, "x is greater than 2", where x is a variable representing a number, is not a proposition, because unless a specific value is given to x we cannot say whether it is true or false.

The Propositional Logic

The propositional logic provides the **connectives** operations $\wedge$, $\vee$, $\sim$, $\rightarrow$, and $\leftrightarrow$ and the two (propositional valuation) constants 'true' and 'false'.

Simple sentences, which are true, or false, are basic propositions. Larger and more complex sentences can be constructed from basic propositions by combining them with connectives. In everyday life, we often combine propositions to form more complex propositions without paying much attention to them.

NOT negation	AND conjunction	OR disjunction	IF_THEN sufficient	ONLY_IF necessary	IF_AND_ONLY_IF equivalent
$\neg, \sim, \bar{}$	$\wedge$	$\vee$	$\rightarrow, \Rightarrow$	$\leftarrow, \Leftarrow$	$\leftrightarrow, \Leftrightarrow$

Let **X** represents proposition "It is raining", and **Y** represents proposition "Joe takes his umbrella". Then [$\sim X$] – negation, [$X \wedge Y$] – conjunction, [$X \vee Y$] – disjunction, [$X \rightarrow Y$] and [$X \leftarrow Y$] – conditional and [$X \leftrightarrow Y$] – equivalence – are propositions.

Exercise. Consider the following "truth tables" for propositions obtained by applying logical operations and understand their meaning.

X	~X
T	F
F	T

X	Y	X∧Y
T	T	T
F	T	F
T	F	F
F	F	F

X	Y	X∨Y
T	T	T
F	T	T
T	F	T
F	F	F

X	Y	X↔Y
T	T	T
F	T	F
T	F	F
F	F	T

X	Y	X→Y
T	T	T
F	T	T
T	F	F
F	F	T

When $A \rightarrow B$ is always true, B follows from A, we express that by $A \Rightarrow B$.

When $A \leftarrow B$ is always true, A follows from B, we express that by $A \Leftarrow B$.

When $A \leftrightarrow B$ is always true, A and B are equivalent, we express that by $A \Leftrightarrow B$

These two conditional claims, "If A, then B" and "A, only if B" refer to two different kinds of conditions: a **sufficient** condition and a **necessary** condition.

A **sufficient** condition is the one that, if satisfied, assures the statement's truth. "If A, then B". If A is truth, then B is also truth, ***A* is sufficient for *B***. If we have A, then we know that B must follow, $A \Rightarrow B$.

Example. Earning a total of 950 points (95%) in English class is a sufficient condition for earning a final grade of A. If you have 950 points, then it follows that you will have a final grade of A. It is not necessary to earn 950 points to earn an A in the English class. You can earn 920 points to earn an A. (We cannot say that if you do not have 950 points then you can't have an A.)

A **necessary** condition of a statement must be satisfied for the statement to be true. "A, only if B" means ***B* is necessary for *A***, $B \Leftarrow A$. If we do not have B, then we will not have A.

Example. I need to put gasoline into my car, without it I will not be able to start the engine. Of course, having gasoline in the car does not guarantee that my car will start. There are many other conditions needed for my car to start, but if there is no gasoline it will definitely not going anywhere.

If A is **sufficient** for B, $A \Rightarrow B$, then B is **necessary** for A.

Is sunlight a necessary or sufficient condition for the roses to bloom?

Is earning a final grade of C a necessary or sufficient condition for passing the course?

Is being a male a necessary or sufficient condition for being a father?

Is attending class regularly and punctually a necessary or sufficient condition for being successful in class?

Is being 20 years old a necessary or sufficient condition for being a college student?

Is completing all the requirements of your degree program a necessary or sufficient condition for earning your degree?

Is being a bird a necessary or sufficient condition for being able to fly?

Definition. A and B are equivalent if both conditions $A \Rightarrow B$ **and** $A \Leftarrow B$ are true. A is necessary **and** sufficient for B, and vice versa. Then we write $A \Leftrightarrow B$.

Definition. The contrapositive of the conditional statement has its antecedent and consequent inverted and flipped: the contrapositive of $A \Rightarrow B$ is $\sim B \Rightarrow \sim A$, or $\sim A \Leftarrow \sim B$. If negation of B is truth, then negation of A will be truth.

Contraposition is a law that says that a conditional statement is logically equivalent to its **contrapositive**, $(A \Rightarrow B) \;\Leftrightarrow\; (\sim B \;\Rightarrow \sim A)$.

Tautologies, Axioms and Inference Rules.

Definition. A proposition F is a **tautology** if it evaluates to truth for all the assignments covering it.

So tautologies are propositional formulae which possess ‘universal logical validity’ and evaluate to true no matter what truth values are assigned to their variables. Examples are

$p \lor (\sim p)$, $q \to (p \to q)$, $p \to (q \to (p \land q))$.

That a proposition is a tautology can be determined by evaluating its possible values, e. g. using the truth tables. Alternatively, a set of tautologies can be adopted as axioms and inference rules, from which all other propositions are derived. The axioms and inference rules can be chosen in many ways. Though not at all the smallest, the following is one possible set, having a familiar and convenient algebraic flavor.

(i) $(p \land q) \leftrightarrow (q \land p)$

(ii) $((p \land q) \land r) \leftrightarrow (p \land (q \land r))$

(iii) $(p \land p) \leftrightarrow p$

(iv) $(p \lor q) \leftrightarrow (q \lor p)$

(v) $((p \lor q) \lor r) \leftrightarrow (p \lor (q \lor r))$

(vi) $(p \lor p) \leftrightarrow p$

(vii) $(\sim(p \land q)) \leftrightarrow ((\sim p) \lor (\sim q))$ – **De Morgan's law 1**

(viii) $(\sim(p \lor q)) \leftrightarrow ((\sim p) \land (\sim q))$ – **De Morgan's law 2**

(ix) $((p \lor q) \land r) \leftrightarrow ((p \land r) \lor (q \land r))$

(x) $((p \land q) \lor r) \leftrightarrow ((p \lor r) \land (q \lor r))$

(xi) $(p \leftrightarrow q) \to ((p \land r) \leftrightarrow (q \land r))$

(xii) $(p \leftrightarrow q) \to ((p \lor r) \leftrightarrow (q \lor r))$

(xiii) $(p \leftrightarrow q) \to ((\sim p) \leftrightarrow (\sim q))$

(xiv) $(p \leftrightarrow q) \to (q \to p)$

(xv) $(p \to q) \leftrightarrow ((\sim p) \lor q)$

(xvi) $(p \leftrightarrow q) \leftrightarrow ((p \to q) \land (q \to p))$

(xvii) $(p \land q) \to p$

(xviii) $(p\leftrightarrow q)\to((q\leftrightarrow r)\to(p\leftrightarrow r))$

(xix) $(p\leftrightarrow q)\to(q\leftrightarrow p)$

(xx) $(p\leftrightarrow p)$

(xxi) $(p \wedge (\sim p))\leftrightarrow$false

(xxii) $(p \vee (\sim p))\leftrightarrow$true

(xxiii) $(\sim(\sim p))\leftrightarrow p$

(xxiv) $(p \wedge \text{true})\leftrightarrow p$

(xxv) $(p \wedge \text{false})\leftrightarrow$false

(xxvi) $(p \vee \text{true})\leftrightarrow$true

(xxvii) $(p \vee \text{false})\leftrightarrow p$

(xxviii) $(\sim\text{true})\leftrightarrow$false

(xxix) $(\sim\text{false})\leftrightarrow$true

(xxx) true

It is easy to verify that all these are in fact tautologies. This large set of axioms needs to be supplemented with only one rule of inference, the **modus ponens** of mediaeval logicians.

<u>**Modus ponens**</u>: From two formulae of the form p and $p \to q$, infer q.

Alternatively, a propositional calculus can be based upon a **set of inference rules**, from which all other statements are derived.

1. <u>Negation introduction</u>: $\{(p \to q), (p \to \sim q)\} \Rightarrow \sim p$. From $(p \to q)$ and $(p \to \sim q)$, infer $\sim p$.
2. <u>Negation elimination</u>: $\{\sim p\} \Rightarrow (p \to q)$. From $\sim p$, infer $(p \to q)$.
3. <u>Double negative elimination</u>: $\sim\sim p \Rightarrow p$. From $\sim\sim p$, infer p.
4. <u>Conjunction introduction</u>: $\{p, q\} \Rightarrow (p \wedge q)$. From p and q, infer $(p \wedge q)$.

5. <u>Conjunction elimination</u>: $(p \wedge q) \Rightarrow p$ and $(p \wedge q) \Rightarrow q$. From $(p \wedge q)$, infer p. From $(p \wedge q)$, infer q.
6. <u>Disjunction introduction</u>: $p \Rightarrow (p \vee q)$ and $q \Rightarrow (p \vee q)$. From p, infer $(p \vee q)$. From q, infer $(p \vee q)$.
7. <u>Disjunction elimination</u>: $\{(p \vee q), (p \to r), (q \to r)\} \Rightarrow r$. From $(p \vee q)$ and $(p \to r)$ and $(q \to r)$, infer r.
8. <u>Biconditional introduction</u>: $\{(p \to q), (q \to p)\} \Rightarrow (p \leftrightarrow q)$. From $(p \to q)$ and $(q \to p)$, infer $(p \leftrightarrow q)$.
9. <u>Biconditional elimination</u>: $(p \leftrightarrow q) \Rightarrow (p \to q)$ and $(p \leftrightarrow q) \Rightarrow (q \to p)$. From $(p \leftrightarrow q)$, infer $(p \to q)$. From $(p \leftrightarrow q)$, infer $(q \to p)$.
10. <u>Modus ponens</u> **(conditional elimination)**: $\{p, (p \to q)\} \Rightarrow q$. From p and $(p \to q)$, infer q.
11. <u>Conditional proof</u> **(conditional introduction)**: $(p \Rightarrow q) \Rightarrow (p \to q)$. From [accepting p allows a proof of q], infer $(p \to q)$.

A list of basic and derived argument forms and logical equivalences is given at the end.

<u>Logical fallacies</u>

A fallacy is reasoning that is evaluated as logically incorrect. Fallacy vitiates the logical validity of the argument and warrants its recognition as unsound.

<u>Formal fallacies</u>

A formal fallacy is an error in logic that can be seen in the argument's form. All formal fallacies are specific types of *non sequiturs* (does not follow).

- Appeal to probability – is a statement that takes something for granted because it would probably be the case (or might be the case).
- Argument from fallacy – also known as fallacy fallacy, assumes that if an argument for some conclusion is fallacious, then the conclusion is false. *If you are paranoid about being stalked does not mean you are not stalked*.
- Base rate fallacy – making a probability judgment based on conditional probabilities, without accounting for the effect of prior probabilities.

- Conjunction fallacy – assumption that an outcome simultaneously satisfying multiple conditions is more probable than an outcome satisfying a single one of them.
- Masked-man fallacy (illicit substitution of identicals) – the substitution of identical designators in a true statement can lead to a false one. *I know how to solve math problems; I don't know whether this is a math problem => I don't know how to solve this problem.*
- Jumping to conclusions – the act of taking decisions without having enough information to be sure they are right.

Propositional fallacies

A propositional fallacy is an error in logic that concerns compound propositions. For a compound proposition to be true, the truth values of its constituent parts must satisfy the relevant logical connectives that occur in it (most commonly: <and>, <or>, <not>, <only if>, <if and only if>). The following fallacies involve inferences whose correctness does not follow from the properties of those logical connectives, and hence, which are not guaranteed to yield logically true conclusions.

- Affirming a disjunct – A or B; A, therefore not B.
- Affirming the consequent –if A, then B; B, therefore A.
- Denying the antecedent – if A, then B; not A, therefore not B.

Quantification fallacies

A quantification fallacy is an error in logic where the quantifiers of the premises are in contradiction to the quantifier of the conclusion.

- Existential fallacy – an argument that has a universal premise and a particular conclusion. "In a communist society everyone has everything (s)he needs", or, "In a communist society everyone suffers from oppression", or, "Every Unicorn has one horn on its forehead".
- A vacuous truth is a conditional statement with a false antecedent. A statement that asserts that all members of the empty set have a certain

property. For example, the statement "all students in the room are in math 9 class" will be true whenever there are no students in the room. In this case, the statement "all students in the room are not in math 9 class" would also be vacuously true, as would the conjunction of the two: "all students in the room are in Math 9 and are not in Math 9”.

Syllogistic fallacies – logical fallacies that occur in syllogisms.

- Affirmative conclusion from a negative premise (illicit negative) – when a categorical syllogism has a positive conclusion, but at least one negative premise. “Smart people don’t eat junk food. I do not eat junk food. Therefore, I a smart”.
- Fallacy of exclusive premises – a categorical syllogism that is invalid because both of its premises are negative.
- Fallacy of four terms (quaternio terminorum) – a categorical syllogism that has four terms. *Nothing is better than eternal happiness; ham sandwich is better than nothing => ham sandwich is better than eternal happiness*.
- Illicit major – a categorical syllogism that is invalid because its major term is not distributed in the major premise but distributed in the conclusion. All A are B; No C are A. Therefore, no C are B.
- Illicit minor – a categorical syllogism that is invalid because its minor term is not distributed in the minor premise but distributed in the conclusion. Pie is good. Pie is unhealthy. Thus, all good things are unhealthy.
- Negative conclusion from affirmative premises (illicit affirmative) – when a categorical syllogism has a negative conclusion but affirmative premises. All A is B. All B is C. Hence, some C is not A.
- Fallacy of the undistributed middle – the middle term in a categorical syllogism is not distributed. All Z is B; All Y is B. Therefore, all Y is Z.
- Modal fallacy – confusing possibility with necessity.

Informal fallacies

Informal fallacies – arguments that are fallacious for reasons other than structural (formal) flaws and usually require examination of the argument's content.

- Appeal to the stone (argumentum ad lapidem) – dismissing a claim as absurd without demonstrating proof for its absurdity.
- …
- Correlation proves causation (post hoc ergo propter hoc)
- Divine fallacy (argument from incredulity) – arguing that, because something is so incredible/amazing/ununderstandable, it must be the result of superior, divine, alien or paranormal agency.
- Double counting – counting events or occurrences more than once in probabilistic reasoning, which leads to the sum of the probabilities of all cases exceeding unity.
- Equivocation – the misleading use of a term with more than one meaning (by glossing over which meaning is intended at a particular time).
- …
- Psychologist's fallacy – an observer presupposes the objectivity of his own perspective when analyzing a behavioral event.
- Red herring – a speaker attempts to distract an audience by deviating from the topic at hand by introducing a separate argument the speaker believes is easier to speak to.
- Referential fallacy – assuming all words refer to existing things and that the meaning of words reside within the things they refer to, as opposed to words possibly referring to no real object or that the meaning of words often comes from how we use them.
- …

A summary of logical equivalences.

Commutative laws:

1. $(A \wedge B) \Leftrightarrow (B \wedge A)$
2. $(A \vee B) \Leftrightarrow (B \vee A)$
3. $(A \Leftrightarrow B) \Leftrightarrow (B \Leftrightarrow A)$

Associative laws:

1. $\big(A \wedge (B \wedge C)\big) \Leftrightarrow \big((A \wedge B) \wedge C\big)$
2. $\big(A \vee (B \vee C)\big) \Leftrightarrow \big((A \vee B) \vee C\big)$
3. $\big(A \Leftrightarrow (B \Leftrightarrow C)\big) \Leftrightarrow \big((A \Leftrightarrow B) \Leftrightarrow C\big)$

Distributive laws:

4. $\big(A \wedge (B \vee C)\big) \Leftrightarrow \big((A \wedge B) \vee (A \wedge C)\big)$
5. $\big(A \vee (B \wedge C)\big) \Leftrightarrow \big((A \vee B) \wedge (A \vee C)\big)$
6. $\big(A \Rightarrow (B \wedge C)\big) \Leftrightarrow \big((A \Rightarrow B) \wedge (A \Rightarrow C)\big)$
7. $\big(A \Rightarrow (B \vee C)\big) \Leftrightarrow \big((A \Rightarrow B) \vee (A \Rightarrow C)\big)$
8. $\big((A \wedge B) \Rightarrow C\big) \Leftrightarrow \big((A \Rightarrow C) \vee (B \Rightarrow C)\big)$
9. $\big((A \vee B) \Rightarrow C\big) \Leftrightarrow \big((A \Rightarrow C) \wedge (B \Rightarrow C)\big)$

Negation laws:

1. $\sim(A \wedge B) \Leftrightarrow \big(\sim(A) \vee \sim(B)\big)$
2. $\sim(A \vee B) \Leftrightarrow \big(\sim(A) \wedge \sim(B)\big)$
3. $\sim(\sim A) \Leftrightarrow A$
4. $\sim(A \Rightarrow B) \Leftrightarrow \big(A \wedge \sim(B)\big)$
5. $\sim(A \Leftrightarrow B) \Leftrightarrow (\sim(A) \Leftrightarrow B)$
6. $\sim(A \Leftrightarrow B) \Leftrightarrow \big(A \Leftrightarrow \sim(B)\big)$

Implication laws:

1. $(A \Rightarrow B) \Leftrightarrow \Big(\sim\big(A \wedge \sim(B)\big)\Big)$
2. $(A \Rightarrow B) \Leftrightarrow (\sim(A) \vee B)$
3. $(A \Rightarrow B) \Leftrightarrow \big(\sim(B) \Rightarrow \sim(A)\big)$
4. $(A \Leftrightarrow B) \Leftrightarrow \big((A \Rightarrow B) \wedge (B \Rightarrow A)\big)$
5. $(A \Leftrightarrow B) \Leftrightarrow \big(\sim(A) \Leftrightarrow \sim(B)\big)$

Table of Basic and Derived Argument Forms ($\vdash$ and $\Rightarrow$ mean "infer", "entail").

Basic and Derived Argument Forms		
Name	Sequent	Description
Modus Ponens	$((p \to q) \wedge p) \vdash q$	If *p* then *q*; *p*; therefore *q*
Modus Tollens	$((p \to q) \wedge \neg q) \vdash \neg p$	If *p* then *q*; not *q*; therefore not *p*
Hypothetical Syllogism	$((p \to q) \wedge (q \to r)) \vdash (p \to r)$	If *p* then *q*; if *q* then *r*; therefore, if *p* then *r*
Disjunctive Syllogism	$((p \vee q) \wedge \neg p) \vdash q$	Either *p* or *q*, or both; not *p*; therefore, *q*
Constructive Dilemma	$((p \to q) \wedge (r \to s) \wedge (p \vee r)) \vdash (q \vee s)$	If *p* then *q*; and if *r* then *s*; but *p* or *r*; therefore *q* or *s*
Destructive Dilemma	$((p \to q) \wedge (r \to s) \wedge (\neg q \vee \neg s)) \vdash (\neg p \vee \neg r)$	If *p* then *q*; and if *r* then *s*; but not *q* or not *s*; therefore not *p* or not *r*
Bidirectional Dilemma	$((p \to q) \wedge (r \to s) \wedge (p \vee \neg s)) \vdash (q \vee \neg r)$	If *p* then *q*; and if *r* then

		s; but *p* or not *s*; therefore *q* or not *r*
Simplification	$(p \land q) \vdash p$	*p* and *q* are true; therefore *p* is true
Conjunction	$p, q \vdash (p \land q)$	*p* and *q* are true separately; therefore they are true conjointly
Addition	$p \vdash (p \lor q)$	*p* is true; therefore the disjunction (*p* or *q*) is true
Composition	$((p \to q) \land (p \to r)) \vdash (p \to (q \land r))$	If *p* then *q*; and if *p* then *r*; therefore if *p* is true then *q* and *r* are true
De Morgan's Theorem (1)	$\neg(p \land q) \vdash (\neg p \lor \neg q)$	The negation of (*p* and *q*) is equiv. to (not *p* or not *q*)
De Morgan's Theorem (2)	$\neg(p \lor q) \vdash (\neg p \land \neg q)$	The negation of (*p* or *q*) is equiv. to

		(not *p* and not *q*)
Commutation (1)	$(p \vee q) \vdash (q \vee p)$	(*p* or *q*) is equiv. to (*q* or *p*)
Commutation (2)	$(p \wedge q) \vdash (q \wedge p)$	(*p* and *q*) is equiv. to (*q* and *p*)
Commutation (3)	$(p \leftrightarrow q) \vdash (q \leftrightarrow p)$	(*p* is equiv. to *q*) is equiv. to (*q* is equiv. to *p*)
Association (1)	$(p \vee (q \vee r)) \vdash ((p \vee q) \vee r)$	*p* or (*q* or *r*) is equiv. to (*p* or *q*) or *r*
Association (2)	$(p \wedge (q \wedge r)) \vdash ((p \wedge q) \wedge r)$	*p* and (*q* and *r*) is equiv. to (*p* and *q*) and *r*
Distribution (1)	$(p \wedge (q \vee r)) \vdash ((p \wedge q) \vee (p \wedge r))$	*p* and (*q* or *r*) is equiv. to (*p* and *q*) or (*p* and *r*)
Distribution (2)	$(p \vee (q \wedge r)) \vdash ((p \vee q) \wedge (p \vee r))$	*p* or (*q* and *r*) is equiv. to (*p* or *q*) and (*p* or *r*)
Double Negation	$p \vdash \neg\neg p$	*p* is equivalent to the negation of not *p*

Transposition	$(p \to q) \vdash (\neg q \to \neg p)$	If *p* then *q* is equiv. to if not *q* then not *p*
Material Implication	$(p \to q) \vdash (\neg p \vee q)$	If *p* then *q* is equiv. to not *p* or *q*
Material Equivalence (1)	$(p \leftrightarrow q) \vdash ((p \to q) \wedge (q \to p))$	(*p* iff *q*) is equiv. to (if *p* is true then *q* is true) and (if *q* is true then *p* is true)
Material Equivalence (2)	$(p \leftrightarrow q) \vdash ((p \wedge q) \vee (\neg p \wedge \neg q))$	(*p* iff *q*) is equiv. to either (*p* and *q* are true) or (both *p* and *q* are false)
Material Equivalence (3)	$(p \leftrightarrow q) \vdash ((p \vee \neg q) \wedge (\neg p \vee q))$	(*p* iff *q*) is equiv to., both (*p* or not *q* is true) and (not *p* or *q* is true)
Exportation[9]	$((p \wedge q) \to r) \vdash (p \to (q \to r))$	from (if *p* and *q* are true then *r* is true) we can prove (if *q* is true then *r* is true, if *p* is

		true)
Importation	$(p \rightarrow (q \rightarrow r)) \vdash ((p \wedge q) \rightarrow r)$	If *p* then (if *q* then *r*) is equivalent to if *p* and *q* then *r*
Tautology (1)	$p \vdash (p \vee p)$	*p* is true is equiv. to *p* is true or *p* is true
Tautology (2)	$p \vdash (p \wedge p)$	*p* is true is equiv. to *p* is true and *p* is true
Tertium non datur (Law of Excluded Middle)	$\vdash (p \vee \neg p)$	*p* or not *p* is true
Law of Non-Contradiction	$\vdash \neg(p \wedge \neg p)$	*p* and not *p* is false, is a true statement