Homework 18: Coordinate geometry review

HW18 is Due on March 3.

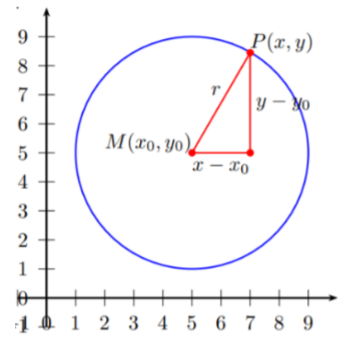
1. Coordinate geometry: Introduction

The **midpoint** M of a segment AB with endpoints A(x_1, y_1) and B(x_2, y_2) has coordinates:

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by the following formula:

$$\boldsymbol{d} = \sqrt{(\boldsymbol{x_2} - \boldsymbol{x_2})^2 + (\boldsymbol{y_2} - \boldsymbol{y_1})^2}$$



Line: $y = \boldsymbol{m}x + \boldsymbol{b}$

with a **slope** $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ and intercept ***b.***

Parabola: $y = ax^2 + bx + c$ (standard form) or $y = a(x-h)^2 + \mathrm{k}$ (vertex form)

Circles: The equation of the circle with the center $M(x_0, y_0)$ and radius ***r*** is: $(x - x_0)^2 + (y - y_0)^2 = r^2$.

2. Graphs of functions

In general, the relation between x and y could be more complicated and could be given by some formula of the form $y = f(x)$, where f is some function of x (i.e., some formula which contains x). Then the set of all points whose coordinates satisfy this relation is called the **graph** of f.

- Vertical shift: $y = f(x) + \boldsymbol{k}$, shift up by k $y = f(x) - \boldsymbol{k}$, shift down by k $\mathrm{k} > 0$
- Horizontal shift: $y = f(x - h)$, shift right by h $y = f(x + h)$, shift to the left by h $h > 0$
- Reflection, x-axis: $y = -f(x)$, multiply the function by -1.
- Reflection, y-axis: $y = f(-x)$, multiply the argument by -1.

3. Quadratic function (revisited +)

Quadratic equation in a standard form: $ax^2 + bx + c = 0$

- a, b, c – coefficients, determinant D: $\boldsymbol{D} = \boldsymbol{b^2} - \boldsymbol{4ac}$, solutions(roots): $\boldsymbol{x_{1,2}} = \frac{-\boldsymbol{b} \pm \sqrt{\boldsymbol{D}}}{\boldsymbol{2a}}$
- **D** determines the number of roots! ($D < 0$ no solutions, $D = 0$ one solution, $D > 0$ two solutions)

Quadratic function in a factored form: $y = a(x - x_1)(x - x_2)$, where

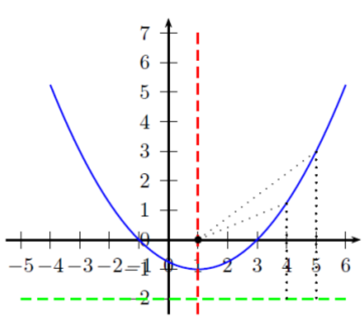
- roots: the numbers x_1 and x_2 – solutions of the quadratic equation ($y = 0$)
- **Vieta's formulas:** The roots are related to the coefficients: $\boldsymbol{x_1 x_2} = \frac{\boldsymbol{c}}{\boldsymbol{a}}$ and $\boldsymbol{x_1} + \boldsymbol{x_2} = -\frac{\boldsymbol{b}}{\boldsymbol{a}}$

Quadratic function in a vertex form: $\boldsymbol{y} = \boldsymbol{a}(\boldsymbol{x} - \boldsymbol{h})^2 + \boldsymbol{k}$

- **Method 1: completing the square**. Use the formulas for fast multiplication.
- **Method 2: find the vertex**. Determine the coefficients a, b, c. Find the vertex x-and y- coordinates
  $\boldsymbol{x_v} = \boldsymbol{h} = -\frac{\boldsymbol{b}}{\boldsymbol{2a}}$. $\boldsymbol{y_v} = \boldsymbol{k} = y(\boldsymbol{x_v}) = a\boldsymbol{x_v}^2 + b\boldsymbol{x_v} + c$

Modified vertex form: rewrite the equation into separate y – and x – part $4\boldsymbol{p}(y - \boldsymbol{k}) = (x - \boldsymbol{h})^2$

Distance from any point on the parabola to focus and directrix: $\boldsymbol{p} = \frac{1}{4a}$
Vertex $V(h, k)$ Focus $F(h, k + \boldsymbol{p})$ directrix $y = k - \boldsymbol{p}$



NEW: Parabola is the set of all points in a plane that are equally distant away from a given point and a given line (see black dotted lines). This given point is called **the focus** (black dot) of the parabola and the line is called **the directrix** (green line).

- If the parabola is of the form $(x - h)^2 = 4p(y - k)$, the vertex is $(h.k)$, the focus is $(h, k + p)$ and directrix is $y = k - p$.

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

ALL GRAPHS/POINTS/FIGURES SHOULD BE DRAWN BY YOU - NOT PRINTED! USE QUADRILE PAPER!

1. For what values of *a* does the polynomial $x^2 + ax + 14$ have no roots? exactly one root? two roots?

2. Let x_1, x_2 be the roots of the equation $x^2 + 3x + 4 = 0$. Without calculating the roots, find:
   a. $x_1^2 + x_2^2$ Hint: use the Vieta's formulas
   b. $\frac{1}{x_1^2} + \frac{1}{x_2^2}$

3. A circle with center (3, 5) intersects the y-axis at (0, 1).
   a. Find the radius of the circle
   b. Find the coordinates of the other point of intersection on the y-axis

4. Convert to vertex form (use completing the square method or find the coordinates of the vertex method) and draw the following graphs:
   a. $y = x^2 - 5x + 5$
   b. $y = x^2 - 4x + 2$
   c. $y = x^2 - x - 1$

5. Convert to vertex form and draw the following graphs. On the graph show the vertex point, the focus point, and the directrix line. You will have to calculate their coordinates/equations first.
   a. $y = -x^2 + 3x - 0.5$
   b. $y = x^2 + 4x - 4$

6. Graph $y = \left(\sqrt{x}\right)^2$. Note that the domain of the function is $x \geq 0$.

7. A triangle ABC has corners A(−3, 0), B(0, 3) and (3, 0). The line $y = \frac{1}{3}x + 1$ separates the triangle in 2. What is the area of the piece lying below the line?