Homework 9: Probability review

HW9 is Due Dec 10.

1. Combinations (binomial coefficients) and permutations.

We discussed binomial coefficients and saw that they provide answers to the following questions:

$nC_k = \binom{n}{k}$ = the number of paths on the chessboard going k units up and n − k to the right
= the number of words that can be written using k ones and n − k zeroes
= the number of ways to choose k items out of n (***order doesn't matter***)

- Formula for binomial coefficients:

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!}$$

- Formula for permutations (the number of ways of choosing k items out of n when ***the order matters***):
Compare it with the number of ways of choosing k items out of n when the order matters:

$$_nP_k = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

Compare the two ways of choosing: the number of ways of choosing k items out of n when the order doesn't matter and when the order matters:

For example, there are $5 \cdot 4 = 20$ ways to choose two items out of 5 if the order matters, and $\frac{5\cdot4}{2} = 10$ if the order does not matter.

2. Binomial probabilities

The binomial coefficients are also useful in calculating probabilities. Imagine that we have some event that happens with probability p ("success") and does not happen with probability $q = 1 - p$ ("failure"). Then the probability of getting k successes in n trials is:

$$P(k\ successes\ in\ n\ trials) = \binom{n}{k} p^k q^{n-k}$$

Where,

n — number of trials; trial: one instance of an experiment. For example, if we are doing a sequence of coin tosses, each coin toss is a trial. If we are shooting ducks, each shot is a trial.

- k — number of successes; <u>success</u>: a trial that ends up in a desired outcome. If we are looking for Heads, success is an outcome of getting a Head. If we are looking at duck shooting, success is a hit.
- n – k — number of failures. <u>failure</u>: a trial that does not end up in a success (missing a duck, getting a Tail while looking for Heads)
- p — probability of success in one try; <u>example</u>: probability of success in one trial (1/6 to roll any number on a die)
- q = 1 – p — probability of failure in one try; <u>example</u>: probability of failure in one trial (1 - 1/6 = 5/6)

Example: You roll a die 100 times. What is the probability of getting a 6 exactly 20 times?

Solution: Here we roll the die n = 100 times, we got a 6 k = 20 times, where the probability for rolling a 6 is p = 1/6, and the probability for not rolling a 6 is q = 5/6. Then using the binomial probability formula, we calculate the probability as:

$$P = \binom{100}{20}\left(\frac{1}{6}\right)^{20}\left(\frac{5}{6}\right)^{80} = \binom{100}{20} \times \frac{5^{80}}{6^{100}}$$

Homework problems

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

Note: In the problems below, calculate the binomial coefficients using the Pascal's triangle, or a calculator.

1. You have 4 letters H and 6 letters T:
   a. How many 10-letter "words" one can write using 4 letters H and 6 letters T?
   b. If we toss a coin 10 times and record the result as a sequence of letters H and T (writing H for heads and T for tails), how many different possible sequences we can get? How many of them will have exactly 6 tails?
   c. If we toss a coin 10 times, what are the chances (the probability) that there will be 6 tails? 3 tails? at least one tails?

2. If we randomly select 100 people form the population of the US, what are the chances that exactly 50 of them will be males? that at least 50 will be males? that all 100 will be males?

3. How many ways are there to divide 12 books
   a. Between two bags [Hints: draw and count. Or use 12 books and 1 divider, in how many ways can you arrange them on a line? This is similar to "stars and bars" method we discussed in class when splitting 5 apples in 2 groups]
   b. Between two bookshelves (order on each bookshelf matters!) (*)
   c. Between three bags
   d. Between three bookshelves (order on each bookshelf matters!) (*)

4. A person is running down the staircase. He is in a rush, so he may jump over some steps. If the staircase is 12 steps (including the top one, where he begins, and the last one, where he ends), in how many ways can he reach the bottom step in 5 jumps? What if there are no restrictions on the number of jumps? [Hint: keep track of the steps he steps on. . .]

5. [You have probably talked about this problem before in Math 6 ... Let's do it again!]
   a. For a group of 25 people, we ask each of them to choose a day of the year (non-leap, so there are 365 possible days). How many possible combinations can we get? Order matters: it is important who had chosen which date.
   b. The same question, but now we additionally require that all chosen dates be different (no repeat, order matters).
   c. In a group of 25 people, what are the chances that no two of them have birthday on the same day? that at least two have the same birthday?