Homework for February 11, 2024

Algebra.

Review the last algebra classwork handouts. Solve the unsolved problems from the previous homeworks. Try solving the following problems.

1. Assume that the set of rational numbers $\mathbb{Q}$ is divided into two subsets, $\mathbb{Q}_<$ and $\mathbb{Q}_>$, such that all elements of $\mathbb{Q}_>$ are larger than any element of $\mathbb{Q}_<$: $\forall a \in \mathbb{Q}_<, \forall b \in \mathbb{Q}_>, a < b$.
   a. Prove that if $\mathbb{Q}_>$ contains the smallest element, $\exists b_0 \in \mathbb{Q}_>, \forall b \in \mathbb{Q}_>, b_0 \leq b$, then $\mathbb{Q}_<$ does not contain the largest element
   b. Prove that if $\mathbb{Q}_<$ contains the largest element, $\exists a_0 \in \mathbb{Q}_<, \forall a \in \mathbb{Q}_<, a \leq a_0$, then $\mathbb{Q}_>$ does not contain the smallest element
   c. Present an example of such a partition, where neither $\mathbb{Q}_>$ contains the smallest element, nor $\mathbb{Q}_<$ contains the largest element
2. Prove the following properties of countable sets. For any two countable sets, A, B,
   a. Union, $A \cup B$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
   b. Product, $A \times B = \{(a, b), a \in A, b \in B\}$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
   c. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 \ldots \cup A_n) = \aleph_0$

3. Let W be the set of all "words" that can be written using the alphabet consisitng of 26 lowercase English letters; by a "word", we mean any (finite) sequence of letters, even if it makes no sense – for example, abababaaaaa. Prove that W is countable. [Hint: for any n, there are only finitely many words of length n.]
4. Compare the following real numbers (are they equal? which is larger?)
   a. 1.33333… = 1.(3) and 4/3
   b. 0.09999… = 0.0(9) and 1/10
   c. 99.9999… = 99.(9) and 100
   d. $\sqrt[2]{2}$ and $\sqrt[3]{3}$
5. Simplify the following real numbers. Are these numbers rational? (hint: you may use the formula for an infinite geometric series).

   a. 1/1.1111…=1/1.1(1)
   b. 2/1.2323…=2/1.23(23)
   c. 3/0.123123…=3/0.123(123)
6. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
   a. 1/8
   b. 2/7
   c. 0.1
   d. $0.33333\ldots = 0.(3)$
   e. $0.13333\ldots = 0.1(3)$
7. Try proving the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

Ordering and comparison.

1. $\forall\, a, b \in \mathbb{R}$, one and only one of the following relations holds
   - $a = b$
   - $a < b$
   - $a > b$
2. $\forall\, a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\, a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\, a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}$, such that $a < nb$

Addition and subtraction.

- $\forall\, a, b \in \mathbb{R}, a + b = b + a$
- $\forall\, a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
- $\forall\, a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
- $\forall\, a \in \mathbb{R}, \exists -a \in \mathbb{R}, a + (-a) = 0$
- $\forall\, a, b \in \mathbb{R}, a - b = a + (-b)$
- $\forall\, a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$

Geometry.

Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework (some problems below are repeated – skip the ones you have already done).

Problems.

1. Review derivation of the equation describing an ellipse and derive in a similar way,
   a. Equation of an ellipse, defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.
   b. Equation of a hyperbola, defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$.
2. Find (describe) set of all points formed by the centers of the circles that are tangent to a given circle of radius r and a line at a distance $d > r$ from its center, O.
3. Using the method of coordinates, prove that the geometric locus of points from which the distances to two given points have a given ratio, $q \neq 1$, is a circle.
4. Find the equation of the locus of points equidistant from two lines, $y = ax + b$ and $y = mx + n$, where a, b, m, n are real numbers.
5. Find the distance between the nearest points of the circles,
   a. $(x-2)^2 + y^2 = 4$ and $x^2 + (y-1)^2 = 9$
   b. $(x+3)^2 + y^2 = 4$ and $x^2 + (y-4)^2 = 9$
   c. $(x-2)^2 + (y+1)^2 = 4$ and $(x+1)^2 + (y-3)^2 = 5$
   d. $(x-a)^2 + y^2 = r_1^2$ and $x^2 + (y-b)^2 = r_2^2$