January 7, 2024

Algebra.

Cartesian product.

Given two sets, A and B, we can construct a third set, C, which is made of all possible ordered pairs of the elements of these sets, (a, b), where $a \in A$ and $b \in B$. We thus have a **binary operation**, which acts on a pair of objects (sets A and B) and returns a third object (set C). Following Rene Descartes, who first considered such construction in the context of Cartesian coordinates of points on a plane, in mathematics such operation is called Cartesian product.

A **Cartesian product** is a mathematical operation that returns a (product) set from multiple sets. For two sets A and B, the Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$,

$$A \times B = \{(a, b): a \in A \wedge b \in B\}$$

Example 1. A table can be created from a single row and a single column, by taking the Cartesian product of a set of objects in a row and a set of objects in a column. In the Cartesian product row × column, the cells of the table contain ordered pairs of the form (row object, column object).

Example 2. Another example is a 52 (or 36) card deck. In a 52 card deck, the standard playing card ranks {A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2} form a 13-element set. The card suits {♠, ♥, ♦, ♣} form a four-element set. The Cartesian product of these two sets returns a 52-element set consisting of 52 ordered pairs, which correspond to all 52 possible playing cards. Ranks × Suits returns a set of the form {(A, ♠), (A, ♥), (A, ♦), (A, ♣), (K, ♠), ..., (3, ♣), (2, ♠), (2, ♥), (2, ♦), (2, ♣)}. Suits × Ranks returns a set of the form {(♠, A), (♠, K), (♠, Q), (♠, J), (♠, 10), ..., (♣, 6), (♣, 5), (♣, 4), (♣, 3), (♣, 2)}. Are these two sets different?

The Cartesian product A × B is **not commutative**, because the elements in the ordered pairs are reversed.

$$\{(a, b): a \in A \wedge b \in B\} = A \times B \neq B \times A = \{(b, a): a \in A \wedge b \in B\}$$

Exercise 1. Construct Cartesian product for sets:

- $A = \{13,14\}$; $B = \{1,1\}$
- $A = \{3,5,7\}$; $B = \{7,5,3\}$
- $A = \{a,b,c,d,e,f,g,h\}$; $B = \{1,2,3,4,5,6,7,8\}$
- $A = \{J,F,M,A,M,J,J,A,S,O,N,D\}$; $B = \{n: n \in \mathbb{N} \wedge n \leq 31\}$

Exercise 2. Check non-commutativity for Cartesian product of sets in Exercise 1 (construct $B \times A$).

Exercise 3. For which particular cases is the Cartesian product commutative?

The Cartesian product is **not associative**,

$$(A \times B) \times C \neq A \times (B \times C)$$

For example, if $A = \{1\}$, then, $(A \times A) \times A = \{((1,1),1)\} \neq \{(1,(1,1))\} = A \times (A \times A)$

Exercise 4. For which particular cases is the Cartesian product associative?

The Cartesian product has the following property with respect to intersections,

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$$

The above statement is not true if we replace intersection with union,

$$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$$

Exercise 5. Prove the following distributivity properties of Cartesian products,

- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

Exercise 6. Present several examples of Cartesian products of sets, which are encountered in everyday life. Describe (define) the constituent sets. Discuss how properties listed above are manifest in the resultant Cartesian products.

Solutions to some homework problems.

1. **Problem.** Using the inclusion-exclusion principle, find how many natural numbers $n < 100$ are not divisible by 3, 5 or 7.

Solution. For $n < 100$, there are 33 divisible by 3, $|A_3| = 33$, 19 divisible by 5, $|A_5| = 19$, 14 numbers divisible by 7, $|A_7| = 14$. Also, there are 6 numbers divisible by $3 \cdot 5 = 15$, 4 divisible by $3 \cdot 7 = 21$, 2 divisible by $5 \cdot 7 = 35$, and none divisible by $3 \cdot 5 \cdot 7 = 105$. Hence, the answer is $99 - |A_3 + A_5 + A_7| = 99 - |A_3| - |A_5| - |A_7| + |A_{3 \cdot 5}| - |A_{3 \cdot 7}| - |A_{5 \cdot 7}| = 99 - (33 + 19 + 14 - 6 - 4 - 2 + 0) = 99 - 54 = 45$.

2. **Problem.** Four letters a, b, c, d, are written down in random order. Using the inclusion-exclusion principle, find probability that at least one letter will occupy its alphabetically ordered place? What is the probability for five letters?

Solution. The set of outcomes where at least one letter occupies its alphabetic place is a complement of the set of derangements of four letters, D_4, to a set of all permutations of the 4 letters, which is also the set of all possible outcomes, $\mathrm{P}_4 = I$. Hence, the number of elements in the set of favorable outcomes where at least one letter is in its proper place, is the difference between the number of elements in the set of all permutations, $|\mathrm{I}| = |\mathrm{P}_4| = 4! = 24!$ and the number of derangements, $\mathrm{d}_4 = |\mathrm{D}_4| = !\,4 = 9$. The corresponding probability is, $\mathrm{p} = \frac{|I| - d_4}{d_4} = \frac{3}{8}$.

3. Using the inclusion-exclusion principle, find the probability that if we randomly write a row of digits from 0 to 9, no digit will appear in its proper ordered position.

Solution. The outcomes where no digit appears in its proper ordered position comprise the set of derangements of 9 objects, D_9. The number of elements in this set is, $\mathrm{d}_9 = |\mathrm{D}_9| = !\,9 = 9! - \binom{9}{1} 8! + \binom{9}{2} 7! - \binom{9}{3} 6! + \binom{9}{4} 5! - \binom{9}{5} 4! + \binom{9}{6} 3! - \binom{9}{7} 2! + \binom{9}{8} 1! - \binom{9}{9} 0! = 133496$. The probability of such a derangement is, $\mathrm{p} = \frac{\mathrm{d}_9}{9!} = \frac{133496}{362880} = 0.36787918871, \frac{1}{\mathrm{p}} = 2.71828369389 \approx \mathrm{e}$.

4. Secretary prepared 5 different letters to be sent to 5 different addresses. For each letter, she prepared an envelope with its correct address. If the 5 letters are to be put into the 5 envelopes at random, what is the probability that
   a. no letter will be put into the envelope with its correct address?
   b. only 1 letter will be put into the envelope with its correct address?
   c. only 2 letters will be put into the envelope with its correct address?
   d. only 3 letters will be put into the envelope with its correct address?
   e. only 4 letters will be put into the envelope with its correct address?
   f. all 5 letters will be put into the envelope with its correct address?

Solution. In each case, the "positive" outcomes comprise set of derangements of 5 objects with the corresponding number of them fixed.

a. $\mathrm{p} = \frac{\mathrm{d}_5}{5!} = \frac{44}{120} = \frac{11}{30} = 0.36(6)$.
b. $p = \frac{\binom{5}{1} d_4}{5!} = \frac{45}{120} = \frac{3}{8} = 0.375.$
c. $p = \frac{\binom{5}{2} d_3}{5!} = \frac{20}{120} = \frac{1}{6} = 0.16(6)$
d. $p = \frac{0}{5!} = 0$
e. $p = \frac{1}{5!} = \frac{1}{120}$

5. If 9 dice are rolled, what is the probability that all 6 numbers appear?

Solution. For that, we might calculate the probability that at least one of the numbers will not appear after $m = 9$ rolls. Let us denote A_n the set of outcomes where a number n ($n = 1, \dots, 6$) does not appear. The number of such outcomes is, $|A_n| = 5^9$. Then, the set of outcomes A where at least one of the numbers does not appear is the union of these sets, $\mathrm{A} = A_1 \cup A_2 \dots \cup A_6$. The total number of outcomes (the number of elements in the universal set) after 9 rolls is 6^9. The probability we are looking for is given by the ratio of the cardinality of set A to that of the universal set, $p_9 = \frac{|A|}{6^9}$. $|A|$ is found using the inclusion-exclusion principle, $|A| = \binom{6}{1} 5^9 - \binom{6}{2} 4^9 + \binom{6}{3} 3^9 - \binom{6}{4} 2^9 + \binom{6}{5} 1^9$, so $1 - p_9 = \frac{6^9 - \binom{6}{1}5^9 + \binom{6}{2}4^9 - \binom{6}{3}3^9 + \binom{6}{4}2^9 - \binom{6}{5}1^9}{6^9} = \frac{1905120}{6^9} = \frac{245}{1296} \approx 0.189.$