October 22, 2023

Geometry.

"Direct" and "Inverse" Theorems.

Each theorem consists of premise and conclusion. Premise is a proposition supporting or helping to support a conclusion.

If we have two propositions, A (premise) and B (conclusion), then we can make a proposition $A \Rightarrow B$ (If A is truth, then B is also truth, A is sufficient for B, or B follows from A, or B is necessary for A). This statement is sometimes called the "direct" theorem and must be proven.

Or we can construct a proposition $A \Leftarrow B$ (A is truth only if B is also truth, A is necessary for B, or A follows from B, B is sufficient for A), which is sometimes called the "inverse" theorem, and also must be proven.

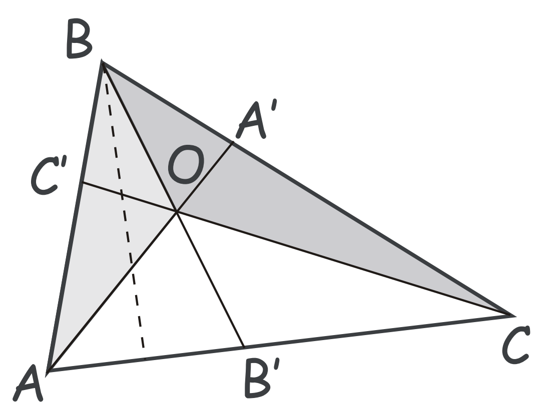
While some theorems offer only necessary or only sufficient condition, most theorems establish equivalence of two propositions, $A \Leftrightarrow B$.

Ceva's Theorem.

Definition. Cevian is a line segment in a triangle, which joins a vertex with a point on the opposite side.

Theorem (Ceva). In a triangle ABC, three cevians AA', BB', and CC' are concurrent (intersect at a single point O) **if and only if**

$$\frac{|AB'|}{|B'C|} \cdot \frac{|CA'|}{|A'B|} \cdot \frac{|BC'|}{|C'A|} = 1$$



This theorem was published by Giovanni Ceva in his 1678 work De lineis rectis.

Direct Ceva's theorem. Geometrical proof.

For the Ceva's theorem the premise (A) is "Three Cevians in a triangle ABC, AA', CC', BB', are concurrent". The conclusion (B) is,

$\frac{|AC'|}{|C'B|} \cdot \frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} = 1$. The full statement of the "direct" theorem is $A \Rightarrow B$, i.e.,

If three cevians in a triangle ABC, AA', CC', BB', are concurrent, **then** $\frac{|AC'|}{|C'B|} \cdot \frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} = 1$ is true. From A follows B, $A \Rightarrow B$. Again, premise in the "direct" theorem provides sufficient condition for the conclusion to hold. Clearly, the conclusion B is the necessary condition for the premise A to hold.

Proof. Consider triangles AOB, BOC and COA. Denote their areas S_{AOB}, S_{BOC}, and S_{COA}. The trick is to express the desired ratios of the lengths of the 6 segments, $|AB'|:|B'C|$, $|CA'|:|A'B|$, $|BC'|:|C'A|$, in terms of the ratios of these areas. We note that some triangles share altitudes. Therefore,

$$\frac{|AB'|}{|B'C|} = \frac{S_{ABB'}}{S_{B'BC}}; \ \frac{|AB'|}{|B'C|} = \frac{S_{AOB'}}{S_{B'OC}}, \text{and so on.}$$

The above two equalities yield,

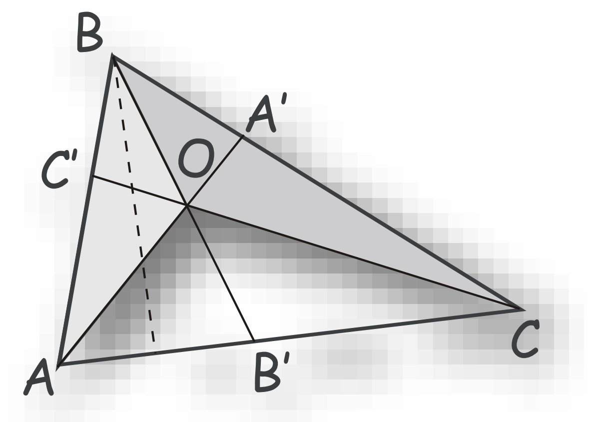
$$\frac{|AB'|}{|B'C|} = \frac{S_{ABB'} - S_{AOB'}}{S_{B'BC} - S_{B'OC}} = \frac{S_{AOB}}{S_{BOC}}$$

Repeating this for the other ratios along the sides of the triangle we obtain,

$$\frac{|AB'|}{|B'C|} \cdot \frac{|CA'|}{|A'B|} \cdot \frac{|BC''|}{|C'A|} = \frac{S_{AOB}}{S_{BOC}} \cdot \frac{S_{AOC}}{S_{BOA}} \cdot \frac{S_{BOC}}{S_{COA}} = 1,$$

which completes the proof.

"Inverse" Ceva's theorem. Geometrical proof.



Let us formulate the "inverse Ceva's theorem", the theorem where premise and conclusion switch places.

If in a triangle ABC three chevians divide sides in such a way that

$$\frac{|AC'|}{|C'B|} \cdot \frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} = 1 \qquad (1)$$

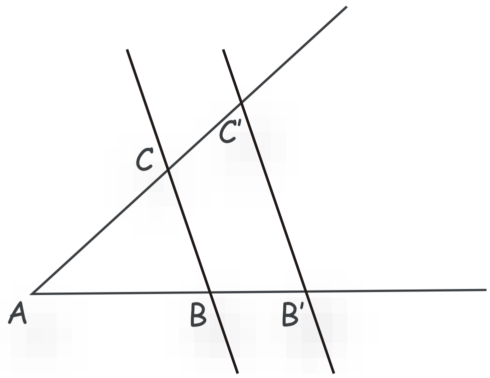
holds, **then** they are concurrent. A follows from B, $B \Rightarrow A$, or $A \Leftarrow B$, or, $\sim A \Rightarrow \sim B$, in other words if the three cevians of a triangle ABC are not concurrent, then $\frac{|AC'|}{|C'B|} \cdot \frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} \neq 1$. Three cevians being concurrent is a necessary condition for the relation $\frac{|AC'|}{|C'B|} \cdot \frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} = 1$ to hold.

Proof. An inverse theorem can often be proven by contradiction (reductio ad absurdum), assuming that it does not hold and arriving at a contradiction with the already proven direct theorem. Assume that Eq. (1) holds, but one of the cevians, say BB', does not pass through the intersection point, O, of the other two cevians. Let us then draw another cevian, BB'', which passes through O. By direct Ceva theorem we have then, $\frac{|CB''|}{|B''A|} = \frac{|C'B|}{|AC'|} \cdot \frac{|A'C|}{|BA'|} = \frac{|CB'|}{|B'A|}$, which means that B' and B''coincide, and therefore AB', must pass through O.

Thus, in the case of Ceva's theorem premise and conclusion (propositions A and B) are equivalent, ($A \Leftrightarrow B$), and we can state the theorem as follows

<u>**Theorem (Ceva).**</u> Three cevians in a triangle ABC, AA', CC', BB', are concurrent, **if and only if** $\frac{|AC'|}{|C'B|} \cdot \frac{|BA'|}{|A'C|} \cdot \frac{|CB'|}{|B'A|} = 1$.

“Inverse” Thales theorem.



The “inverse” Thales theorem states

If lengths of segments in the Figure on the left satisfy $\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}$, **then** lines BC and BC' are parallel. The proof is similar to the proof of Ceva’s “inverse” theorem, by assuming the opposite and obtaining a contradiction.

If a theorem establishes the equivalence of two propositions A and B, $A \Leftrightarrow B$, it is actually often the case that the proof of the necessary condition, $A \Leftarrow B$, i. e. the “inverse” theorem, is much simpler than the proof of the “direct” proposition, establishing the sufficiency, $A \Rightarrow B$. It often could be achieved by using the sufficiency condition which has already been proven, and employing the method of “proof by contradiction”, or another similar construct.

<u>**Examples of necessary and sufficient statements**</u>

- Predicate A: “quadrilateral is a square”

  Predicate B: “all four its sides are equal”

  Which of the following holds: $A \Rightarrow B$, $A \Leftarrow B$, $A \Leftrightarrow B$?

  Is A necessary or sufficient condition for B?

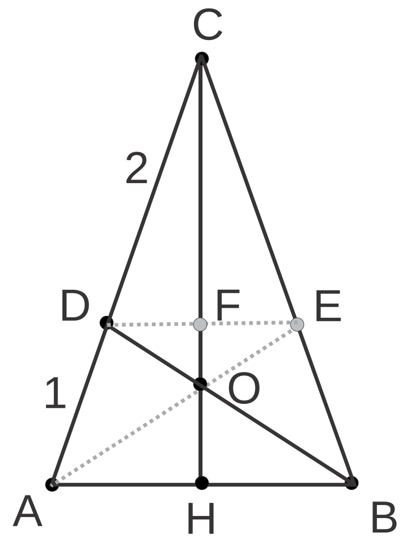
  If a quadrilateral is not square its four sides are not equal. Truth or not? ($A \Leftarrow B$ or $\sim A \Rightarrow \sim B$).

- Predicate A:

  Predicate B:

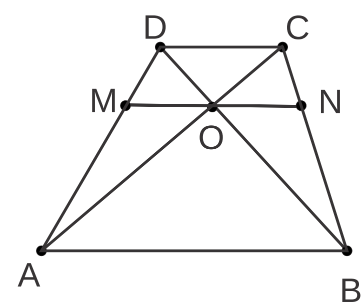
  Which of the following holds: $A \Rightarrow B$, $A \Leftarrow B$, $A \Leftrightarrow B$?

Homework review: problems on similar triangles.



Problem 1 (homework problem #3). In the isosceles triangle ABC point D divides the side AC into segments such that $|AD|:|CD| = 1:2$. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio $|OH|$ to $|CH|$.

Solution. First, let us perform a supplementary construction by drawing the segment DE parallel to AB, $DE||AB$, where point E belongs to the side CB, and point F to DE and the altitude CH. Notice the similar triangles, $AOH \sim DOF$, which implies, $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|}$. By Thales theorem, $\frac{|AH|}{|DF|} = \frac{|AC|}{|AD|} = 1 + \frac{|CD|}{|AD|} = \frac{3}{2}$, and $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|} = \frac{2}{3}$, so that $\frac{|FH|}{|OH|} = \frac{|FO|+|OH|}{|OH|} = \frac{5}{3}$. $\frac{|CH|}{|OH|} = \frac{|CH|}{|FH|}\frac{|FH|}{|OH|} = 3 \cdot \frac{5}{3} = 5$, because $\frac{|CH|}{|FH|} = 1 + \frac{|CF|}{|FH|} = 1 + \frac{|CD|}{|DA|}$. Therefore, the sought ratio is, $\frac{|OH|}{|CH|} = \frac{1}{5}$.



Problem 2 (homework problem #4). In a trapezoid $ABCD$ with the bases $|AB| = a$ and $|CD| = b$, segment MNparallel to the bases, $MN||AB$, connects the opposing sides, $M \in [AD]$ and $N \in [BC]$. MN also passes through the intersection point O of the diagonals, AC and BD, as shown in the Figure. Prove that $|MN| = \frac{2ab}{a+b}$.

Solution. By Thales theorem applied to vertical angles AOB and DOC and parallel lines AB and CD, $\frac{|AM|}{|MD|} = \frac{|BN|}{|NC|} = \frac{|AB|}{|DC|} = \frac{a}{b}$. Consequently, $\frac{|AD|}{|MD|} = \frac{|AM|+|MD|}{|MD|} = \frac{a}{b} + 1 = \frac{|BN|+|NC|}{|NC|} = \frac{|BC|}{|NC|}$. Now, applying the same Thales theorem to angles ADB and ACB and parallel lines MN and AB, we obtain, $\frac{|MO|}{|AB|} = \frac{|MD|}{|AD|} = \frac{1}{\frac{a}{b}+1}$ and $\frac{|ON|}{|AB|} = \frac{|NC|}{|BC|} = \frac{1}{\frac{a}{b}+1}$. Hence, $\frac{|MO|}{|AB|} + \frac{|ON|}{|AB|} = \frac{|MN|}{|AB|} = \frac{2}{\frac{a}{b}+1}$, and $|MN| = \frac{2ab}{a+b}$.