MATH 8: HANDOUT 26 [APR 14, 2024]
NUMBER THEORY 4: CONGRUENCES

REMINDER: EUCLID'S ALGORITHM

Recall that as a corollary of Euclid's algorithm we have the following result:

Theorem. *An integer m can be written in the form*

$$m = ax + by$$

if and only if m is a multiple of $\gcd(a, b)$.

For example, if $a = 18$ and $b = 33$, then the numbers that can be written in the form $18x + 33y$ are exactly the multiples of 3.

To find the values of x, y, one can use Euclid's algorithm; for small a, b, one can just use guess-and-check.

CONGRUENCES

In many situation, we are mostly interested in remainder upon division of different numbers by same integer n. For example, in questions related to the last digit of a number k, we are really looking at remainder upon division of k by 10.

This motivates the following definition: we will write

$$a \equiv b \mod m$$

(reads: a is *congruent* to b modulo m) if a, b have the same remainder upon division by m (or, equivalently, if $a - b$ is a multiple of m).

Congruences can be added and multiplied in the same way as equalities: if

$$a \equiv a' \mod m$$
$$b \equiv b' \mod m$$

then

$$a + b \equiv a' + b' \mod m$$
$$ab \equiv a'b' \mod m$$

Here are some examples:

$$2 \equiv 9 \equiv 23 \equiv -5 \equiv -12 \mod 7$$

$$10 \equiv 100 \equiv 28 \equiv -8 \equiv 1 \mod 9$$

Note: we will occasionally write $a \mod m$ for remainder of a upon division by m.

Since $23 \equiv 2 \mod 7$, we have

$$23^3 \equiv 2^3 \equiv 8 \equiv 1 \mod 7$$

And because $10 \equiv 1 \mod 9$, we have

$$10^4 \equiv 1^4 \equiv 1 \mod 9$$

One important difference is that in general, one can not divide both sides of an equivalence by a number: for example, $5a \equiv 0 \mod m$ does not necessarily mean that $a \equiv 0 \mod m$ (see problem 3b below).

PROBLEMS

1. (a) Use $10 \equiv -1 \mod 11$ to compute $100 \mod 11$; $100,000,000 \mod 11$. Can you derive the general formula for $10^n \mod 11$?

(b) Without doing long division, compute $1375400 \mod 11$. [Hint: $1375400 = 10^6 + 3\cdot 10^5 + 7\cdot 10^4 \dots$]

2. (a) Compute remainders modulo 12 of 5, 5^2, 5^3, ... Find the pattern and use it to compute $5^{1000} \mod 12$

(b) Prove that for any a, m, the following sequence of remainders mod m:
$a \mod m$, $a^2 \mod m$, ...
sooner or later starts repeating periodically (we will find the period later). [Hint: have you heard of pigeonhole principle?]

(c) Find the last digit of 7^{2024}

3. (a) For of the following equations, find at least one integer solution (if exists; if not, explain why)

$$5x \equiv 1 \mod 19$$
$$9x \equiv 1 \mod 24$$
$$9x \equiv 6 \mod 24$$

[Hint: $5x \equiv 1 \mod 19$ is the same as $5x = 1 + 19y$ for some integer y.]

(b) Give an example of a, m such that $5a \equiv 0 \mod m$ but $a \not\equiv 0 \mod m$

4. (a) Show that the equation $ax \equiv 1 \mod m$ has a solution if and only if $\gcd(a, m) = 1$. Such an x is called the *inverse* of a modulo m. [Hint: use Euclid's algorithm, linear combination of a, m equal to 1, and proof by contradiction]

(b) Find the following inverses
inverse of $2 \mod 5$
inverse of $5 \mod 7$
inverse of $7 \mod 11$
Inverse of $11 \mod 41$

5. (a) Find $\gcd(48, 39)$

(b) Solve $48x + 39y = 3$

(c) Find inverse of 39 mod 48.

6. (a) Integers a, b are such that $a^2 + b^2$ is divisible by 3. Show that then $a^2 + b^2$ is divisible by 9.

(b) Integers a, b are such that $a^2 + b^2$ is divisible by 21. Show that then $a^2 + b^2$ is divisible by 441.

***7.** Prove that no positive integer solutions exist for the following equations.

(a) $x^3 = x + 10^n$ [Hint: see if you can prove that $x^3 \equiv x$ mod 3]

(b) $x^3 + y^3 = x + y + 10^n$

***8.** For a positive number n, let $\sigma(n)$ (this is Greek letter "sigma") be the sum of all divisors of n (including 1 and n itself).

Compute
$\sigma(10)$
$\sigma(77)$
$\sigma(p^a)$, where p is prime (the answer, of course, depends on p, a)
$\sigma(p^a q^b)$, where p, q are different primes
$\sigma(10000)$
$\sigma(p_1^{a_1} p_2^{a_2} \dots p_k^{a_k})$, where p_i are distinct primes.