MATH 7: HANDOUT 13
QUADRATIC FORMULA

QUADRATIC EQUATIONS.

Today we discussed how one solves quadratic equation:

$$ax^2 + bx + c = 0$$

The method used is called "completing the square". Here is an example how it works:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 7 = (x+3)^2 - 7 = (x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$$

thus, $x^2 + 6x + 2 = 0$ if and only if $x + 3 + \sqrt{7} = 0$, which gives $x = -3 - \sqrt{7}$, or $x + 3 - \sqrt{7} = 0$, which gives $x = -3 + \sqrt{7}$.

The same trick works in general: if $a = 1$, then

$$\begin{aligned} x^2 + bx + c &= x^2 + 2\tfrac{b}{2}x + c = \left(x^2 + 2\tfrac{b}{2}x + \tfrac{b^2}{2^2}\right) - \tfrac{b^2}{2^2} + c \\ &= \left(x + \tfrac{b}{2}\right)^2 - \tfrac{b^2 - 4c}{4} = \left(x + \tfrac{b}{2}\right)^2 - \tfrac{D}{4} \end{aligned} \tag{1}$$

where $D = b^2 - 4c$.

Thus, $x^2 + bx + c = 0$ is equivalent to

$$\left(x + \tfrac{b}{2}\right)^2 = \tfrac{D}{4}$$

If a is not equal to 1, the answer is similar: $ax^2 + bx + c = 0$ is equivalent to

$$\left(x + \tfrac{b}{2a}\right)^2 = \tfrac{D}{4a^2}, \qquad D = b^2 - 4ac$$

Therefore, if $D < 0$, there are no solutions; if $D \geq 0$, solutions are

$$\begin{aligned} x + \tfrac{b}{2a} &= \pm\frac{\sqrt{D}}{2a} \\ x &= \frac{-b \pm \sqrt{D}}{2a} \end{aligned} \tag{2}$$

HOMEWORK

1. Solve the following equations. Carefully write all the steps in your argument.

(a) $x^2 - 5x + 5 = 0$
(b) $x^2 = 1 + x$
(c) $2x(3 - x) = 1$
(d) $x^3 + 4x^2 - 45x = 0$
(e) $\dfrac{x}{x-2} = x - 1$

2. In the 12th century, Indian mathematician Bhaskara formulated the following problem. Solve it!
Out of a party of monkeys, the square of one fifth of their number diminished by three went into a cave. The one remaining monkey was climbing up a tree. What is the total number of monkeys?

3. (a) Use formula (1) to prove that for any x, $x^2 + bx + c \geq -D/4$, with equality only if $x = -b/2$.
(b) Find the minimal possible value of the expression $x^2 + 4x + 2$
(c) Given a number $a > 0$, find the maximal possible value of $x(a - x)$ (the answer will depend on a).

4. If $x + \frac{1}{x} = 7$, find $x^2 + \frac{1}{x^2}$; $x^3 + \frac{1}{x^3}$.

***5.** Consider the sequence $x_1 = 1$, $x_2 = \frac{x_1}{2} + \frac{1}{x_1}$, $x_3 = \frac{x_2}{2} + \frac{1}{x_2}$ .... Compute the first several terms; does it seem that the sequence is increasing? decreasing? approaching some value? If so, can you guess this value? [Hint: solve equation $x = \frac{x}{2} + \frac{1}{x}$.]