MATH 7: HANDOUT 8
INTRODUCTION TO COMBINATORICS.

COUNTING

Fundamental Principle of Counting (Multiplication rule). If the first task can be performed in m ways, and for each of these a second task can be performed in n ways, and for each combination a third task can be performed in k ways, etc. then this entire sequence of tasks can be performed in $m \cdot n \cdot k \dots$ ways.

Permutations: the choice of k things from a set of n things without repetition ("replacement") and where the **order matters**.

1. Picking first, second, and third place winners from a group. If a group has n members, then this can be done in $n(n-1)(n-2)$ ways.

Permutations of n things: The number of permutations of all n different things: $n!$

1. Arranging/ordering all n members of a group can be done in $n!$ ways.
2. Listing the favorite deserts in the order of choices: if there are n desserts in total, there are $n!$ ways to arrange them in the order of preference.

Combinations: the choice of k things from a set of n things without repetition ("replacement") and where **order does not matter**. Combinations are harder to count: we will talk about it later!

1. Picking three team members from a group (it doesn't matter who is chosen first, or second, or third).
2. Picking two deserts from a tray (the order in which you eat them doesn't matter!).

We call numbers of combination **binomial coefficients**. We can see that they provide answers to the following questions:

$$\begin{aligned}\binom{n}{k} &= \text{rhe number of words that can be written using } k \text{ ones and } n-k \text{ zeroes} \\ &= \text{the number of ways to choose } k \text{ items out of } n \text{ (order doesn't matter)}\end{aligned}$$

FORMULA FOR BINOMIAL COEFFICIENTS

It turns out that there is an explicit formula for $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

Compare it with the number of ways of choosing k items out of n when the order matters:

$${}_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

For example, there are $5 \cdot 4 = 20$ ways to choose to items out of 5 if the order matters, and $\frac{5\cdot 4}{2} = 10$ if the order doesn't matter.

HOMEWORK

1. A dinner in a restaurant consists of 3 courses: appetizer, main course, and dessert. There are 5 possible appetizers, 6 main courses and 3 desserts. How many possible dinners are there?

2. How many ways are there to seat 5 students in a class that has 5 desks? if there are 10 desks?

3. How many ways are there to select first, second and third prize winner if there are 14 athletes in a competition?

4. How many ways are there to put 8 rooks on a the chessboard so that no one attacks the others?

5. A dressmaker has two display windows. The left display is for evening dresses and the one in the right window for regular day dresses. Assuming she can put 10 evening dresses in any order, and separately, 5 regular dresses in any order, how many total possibilities of arranging the two display windows are there?

6. The guidelines at a certain college specify that for the introductory English class, the professor may choose one of 3 specified novels, and choose two from a list of 5 specified plays. Thus, the reading list for this introductory class must have one novel and two plays. How many different reading lists could a professor create within these parameters?

7. (a) There are 15 students in a soccer club. The coach needs to select 11 of them to form the team for a match against another club. How many possibilities does he have?
(b) There are 15 students in a soccer club. The coach needs to select a goalkeeper and 10 players to form the team for a match against another club. How many possibilities does he have? (The difference between two parts is that in the first case, the coach needs to select 11 players –0 no need to specify their positions. In the second part, he needs to select 11 players and specify which of them will be the goalkeeper.)

8. In one of the lotteries run by New York State, "Sweet Million", they randomly choose 6 numbers out of numbers 1–40. If you guess all 6 correctly (order does not matter), you win $1,000,000. [There are also smaller prizes for guessing 5 out of 6, etc., but let us ignore them for now.]
(a) How many ways are there to choose 6 numbers out of 40?
(b) What are your chances of winning?
(c) If a lottery ticket cost $1, how much money does New York State make for each ticket sold (on average)?
(d) **Bonus question:** find online the rules for another NY lottery, "Mega Millions," and analyze your chances to win.

9. In poker, players are drawing "hands" (combinations of 5 cards) from the 52-card deck (4 suits, 13 cards in each).
(a) How many possible hands are there?
(b) What are your chances of drawing a hand in which all cards are spades?
(c) What are your chances of drawing a hand which has 4 queens in it? [Hint: how many such hands are there?]
(d) What are your chances of drawing a royal flush (Ace, King, Queen, Jack, 10 — all of the same suit)? [**Hint:** what are your chances of drawing a royal flush in a given suit, say spades?]