HANDOUT 12: FACTOR!

An equation $f(x) = 0$ can be easily solved if we can write $f(x)$ as a product of simpler functions: $f(x) = f_1(x)f_2(x)$ (this is called *factoring* $f(x)$). Indeed, as discussed last time, in this case $f_1(x)f_2(x) = 0 \iff (f_1(x) = 0) \vee (f_2(x) = 0)$. For example, we can easily solve equation $x^2 - 3x + 2 = 0$ if we notice that $x^2 - 3x + 2 = (x - 1)(x - 2)$ (check!), so $x^2 - 3x + 2 = 0 \iff (x - 1 = 0) \vee (x - 2 = 0)$. Thus, the solutions are $\{1, 2\}$.

In a similar way, factoring also gives an easy way to solve inequalities. For example, to solve $x^2-3x+2 > 0$, notice that $x^2 - 3x + 2 = (x - 1)(x - 2)$ and thus the sign of $x^2 - 3x + 2$ is determined by the signs of $x - 1$ and $x - 2$. Thus, we can consider separately the three regions into which points 1, 2 divide the real line:

- For $x \in (-\infty, 1)$ (that is, $x < 1$), we have $x - 1 < 0$, $x - 2 < 0$, so $(x - 1)(x - 2) > 0$. Thus, any $x < 1$ is a solution of $x^2 - 3x + 2 > 0$.
- For $x \in (1, 2)$ (that is, $1 < x < 2$), we have $x - 1 > 0$, $x - 2 < 0$, so $(x - 1)(x - 2) < 0$. Thus, $x^2 - 3x + 2 > 0$ has no solutions in $(1, 2)$.
- For $x \in (2, \infty)$ (that is, $x > 2$), we have $x - 1 > 0$, $x - 2 > 0$, so $(x - 1)(x - 2) > 0$. Thus, any $x > 2$ is a solution of $x^2 - 3x + 2 > 0$.

We have not yet considered the points $1, 2$ themselves; one easily sees that for $x = 1$ or 2, $(x - 1)(x - 2) = 0$, so these points are not solutions of $(x - 1)(x - 2) > 0$

Thus, the set of solutions is $(-\infty, 1) \cup (2, \infty)$.

The main difficulty is finding a factorization: how did we guess that $x^2 - 3x + 2 = (x - 1)(x - 2)$? We will discuss general rule later; for now, here are some facts that can be useful (each is easy to check by direct calculation)

$$a^2 - b^2 = (a - b)(a + b). \tag{1}$$

In particular, it shows that $x^2 - a^2 = (x - a)(x + a)$, so

$$x^2 - a^2 = 0 \iff (x = a) \vee (x = -a). \tag{2}$$

Another useful fact is

$$(x + a)(x + b) = x^2 + (a + b)x + ab. \tag{3}$$

Thus, to factor $x^2 + 5x + 6$, we need to find a, b such that $(x + a)(x + b) = x + (a + b)x + ab = x^2 + 5x + 6$, so we need $a + b = 5$, $ab = 6$. In this case, one can easily guess the answers: $a = 2, b = 3$.

1. Show that for $D \geq 0$, $x^2 = D \iff (x = \sqrt{D}) \vee (x = -\sqrt{D})$, and for $D < 0$, equation $x^2 = D$ has no solutions.

2. Solve the equation $(x - 1)^2 = 6$.

3. (a) Solve the equation $x^4 - 1 = 0$ (hint: $x^4 = (x^2)^2$).
(b) Solve the inequality $x^4 - 1 > 0$.

4. Solve the following equations. Carefully write all the steps in your argument. Please do not use calculators.

(a) $x^2 - 5x + 4 = 0$ (b) $\dfrac{x}{x-2} = x - 2$ (c) $x^2 = (1 - x)^2$
(d) $x^3 + 3x^2 + 2x = 0$ (e) $x^4 - 5x^2 + 4 = 0$ (f) $x^2 - 6x + 9 = 0$

5. Solve the following inequalities. Carefully write all the steps in your argument. Please do not use calculators.

(a) $x^2 - 5x + 4 < 0$ (b) $x^2 - 5x + 4 > 0$ (c)* $\dfrac{x}{x-2} > x - 2$
(d) $x^3 + 3x^2 + 2x < 0$ (e) $x^2 - 6x + 9 > 0$

6. (a) Factor $x^2 - 2x + 1$
(b) Show that for any $x > 0$, we have $x + \frac{1}{x} \geq 2$.