Math 7: Handout 8
Geometric Sequences.

Geometric Sequences

A sequence of numbers is a geometric sequence if the next number in the sequence is the current number times a constant, called the **common ratio** q. For example, let's consider the sequence: 6, 12, 24, 48, ... Notice that the common ratio is q =2. The first term in the sequence is b_1 =6, the second is $b_2 = 6 \cdot 2 = 12$, $b_3 = 12 \cdot 2$. What is the n^{th} term? For example what is b_{10}?

$b_1 = 6$
$b_2 = 6 \cdot 2 = 12$
$b_3 = (6 \cdot 2) \cdot 2 = 6 \cdot 2^2 = 24$
$b_4 = (6 \cdot 2^2) \cdot 2 = 6 \cdot 2^3 = 48$
...
In general, the n^{th} term $b_n = b_1 \cdot q^{n-1}$

Geometric Mean

A property of a geometric sequence is that any term is the geometric mean of its neighbors. For example, $b_2 = \sqrt{b_1 \cdot b_3} = \sqrt{6 \cdot 24} = 12$. In general,

$$b_n = \sqrt{b_{n-1} \cdot b_{n+1}}$$

Sum of a Geometric Sequence

Let's try to sum $S = 1 + 2 + 4 + ... + 64$. Notice that $2S = 2 + 4 + 8 + ... + 128$; subtract the original sum to get $2S - S = 128 - 1$ (everything else cancels out). Thus $S = 127$. What did we do here? In the geometric sequence 1, 2, ..., 64, the common ratio is $q = 2$. We multiplied S by q=2, which lined up the terms of the sequence to the next term over.
Let's do this for the general case. Let $b_1, ... \, b_n$ be a geometric sequence with common ratio q. Then,

$$S = b_1 + b_2 + ... + b_n$$

$$q \cdot S = q \cdot b_1 + q \cdot b_2 + ... + q \cdot b_n = b_2 + b_3 + ... + b_{n+1}$$
$$q \cdot S - S = b_{n+1} - b_1 = b_1 \cdot q^n - b_1$$
$$S \cdot (q - 1) = b_1 \cdot (q^n - 1)$$
$$S = b_1 \cdot \frac{q^n - 1}{q - 1}$$

Homework

1. Write out the first four terms of each of the following geometric sequences, given the first term b_1 and common ration q.
   a. $b_1 = 1$ and $q = 3$
   b. $b_1 = 1$ and $q = \frac{1}{2}$
   c. $b_1 = -10$ and $q = \frac{1}{2}$
   d. $b_1 = 27$ and $q = -\frac{1}{3}$
2. Calculate $S = 1 + 3 + 9 + 27 + 81 + 243$, first via the method of multiplying by the common ratio, then by plugging into the formula directly. Which method do you like better?
3. Calculate $S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$, using your preferred method.
4. What are the first two terms of the geometric sequence $b_1, b_2, 24, 36, 54 \ldots$? Remember that you can find the common ratio by dividing a term by the previous term.
5. What is the geometric mean of 12 and 3?
6. What is the common ratio of the geometric sequence $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \ldots$? What is b_{10}? b_{99}? b_{100}?
7. Calculate: $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \ldots + \dfrac{1}{2^{10}}$
8. Calculate the sum $1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + \ldots - 2^{15}$
9. The 3-rd term of an arithmetic progression is equal to 1. The 10-th term of it is three times as much as the 6-th term. Find the first term and the common difference. (**Hint:** Use the formula for the n-th term of the progression and write what is given in the problem using this formula.)
10. There are 25 trees at equal distances of 5 meters in a line with a well, the distance of the well from the nearest tree being 10 meters. A gardener waters all trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardener will cover in order to water all the trees.
11. An arithmetic progression has first term $a_1 = a$ and common difference $d = -1$. The sum of the first n terms is equal to the sum of the first $3n$ terms. Express a in terms of n.
12. In the figure shown, point O is the center of the circle. One vertex of the square lies on the circle, and the opposite vertex is point O. If the area of the shaded region is $36\pi - 18$, what is the perimeter of the square?