MATH 6: ASSIGNMENT 16

February 11, 2024

Geometric Sequence

A sequence of numbers is a geometric progression if the next number in the sequence is the current number times a constant called the common ratio, let's call it q. For example, let's consider the sequence:

$$6, 12, 24, 48, ...$$

The first term in the sequence is $b_1 = 6$, the second is $b_2 = 6 \times 2 = 12$, and so on.

The common ratio is $q = 2$. Indeed $b_3 = b2 \times q = 12 \times 2 = 24$ and $b_4 = 24 \times 2 = 48$.

What is the n^{th} term? For example what is b_{10}?

$b_1 = 6$

$b_2 = b_1 \times q = 6 \times 2 = 12$

$b_3 = b_2 \times q = (b_1 \times q) \times q = b_1 \times q^2 = 6 \times 2^2 = 24$

$b_4 = b_3 \times q = (b_1 \times q^2) \times q^2 = b_1 \times q^3 = 6 \times 2^3 = 48$

....

$b_n = b_1 \times q^{n-1}$

So $b_{10} = b_1 \times q^9 = 6 \times 2^9 = 6 \times 512 = 3072$

Sum of a geometric progression

There is a formula for the sum of a geometric progression:

$$S = b_1 + b_2 + b_3 + \cdots + b_n = b_1 \times \frac{(1-q^n)}{1-q}$$

To prove this, we write the sum and we multiply it by q:

$$S = b_1 + b_2 + b_3 + \cdots b_{n-1} + b_n$$

$$qS = qb_1 + qb_2 + qb_3 + \cdots + qb_{n-1} + qb_n$$

Remember that $qb_{n-1} = b_n$, so that the last term is $qb_n = q \times (b_1 \times q^{n-1}) = b_1 \times q^n$.

$$qS = b_2 + b_3 + b_4 + \cdots b_n + b_1 q^n$$

We subtract S from each side:

$$qS - S = b_2 + b_3 + \cdots + b_n + b_1 q^n - (b_1 + b_2 + b_3 + \cdots b_{n-1} + b_n)$$

All terms cancel, except $b_1 q^n$ and b_1 so that:

$$qS - S = b_1 q^n - b_1$$

$$(1-q)S = b_1 q^n - b_1$$

$$S = \frac{b_1 q^n - b_1}{1-q} = \frac{b_1(1-q^n)}{1-q}$$

Homework

1. Write the first 5 terms of a geometric progression if $b_1 = -20$ and $q = \frac{1}{2}$

2. What are the first 2 terms of the geometric progression: b_1, b_2, 24, 36, 54, ...?

3. What is the common ratio of the geometric progression: $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$, ...? What is b_{10}? What is b_{100}?

4. Simplify:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{10}}$$

5. What is the sum: $1 - 2 + 2^2 - 2^3 + 2^4 - 2^5 + \cdots - 2^{15}$?

6. What is the sum: $1 + x + x^2 + x^3 + x^4 + x^5 + \cdots + x^{100}$?

7. A geometric progression has 99 terms, the first term is 12 and the last term is 48. What is the 50th term?

8. If we put one grain of wheat on the first square of the chessboard, two on the second, then four, eight,. . . , approximately how many grains of wheat will there be? (You can use $2^{10} = 1024 \approx 10^3$). Can you estimate the total volume of all this wheat? Compare with the annual wheat harvest of the US, which is about 2 billion bushels. (A grain of wheat is about 10 mm^3; a bushel is about 35 liters, or 0.035 m^3)

9. How many multiples of 7 are there between 1 and 1000? Can you find the sum of them all?

10. Find the sum $1 + 3 + 5 + \cdots + 999$.