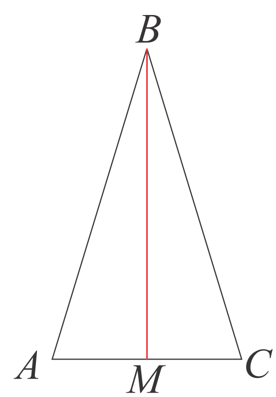
Math 5a. Classwork16.

Theorem. In isosceles triangle the bisector passed to the base (in isosceles triangle the base is the side different from two equal sides) is a median and an altitude as well.

Let the triangle ΔABC be an isosceles triangle, such that $AB = BC$, and BM is a bisector. We need to prove that BM is a median and an altitude, which means that $AM = MC$ and angle $\angle BMC$ is a right angle.



BM is a bisector, so $\angle ABM$ and $\angle MBC$, the triangle ΔABM is an isosceles triangle, so $AB = BC$ and the segment MB is common side for triangles ΔABM and ΔMBC. Based on the Side-Angle-Side criteria, the triangles ΔABM and ΔMBC are congruent. Therefore, $AM = MC$ (BM is a median), angles $\angle A$ and $\angle C$ are congruent. (Isosceles triangle has equal angles adjacent to the base).

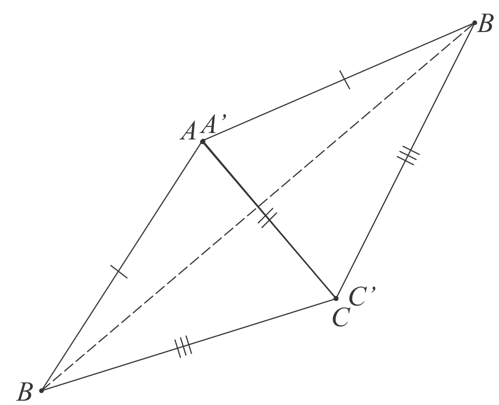
$\angle A + \angle B + \angle C = 180° = 2\angle A + \angle B \Rightarrow 90° = \angle A + \frac{1}{2}\angle B$ but for the triangle ABM (as well as for MBC), $\angle A + \frac{1}{2}\angle B + \angle BMA = 180°$, therefore $\angle BMA = 90°$and BM is also an altitude.

SSS (Side-Side-Side): If three pairs of sides of two triangles are equal in length, then the triangles are congruent.

Let ΔABC and ΔA'B'C' be two triangles such that

$AC = A'C', AC = A'C', BC = B'C'.$

It is required to prove that triangles are congruent. Proving this test by superimposing, the same way as we proved the first two tests, turns out to be awkward, because knowing nothing about the measure of angles, we would not be able co conclude from coincidence of two corresponding sides the other side coincide as well. Instead of superimposing, let us apply *juxtaposing*.



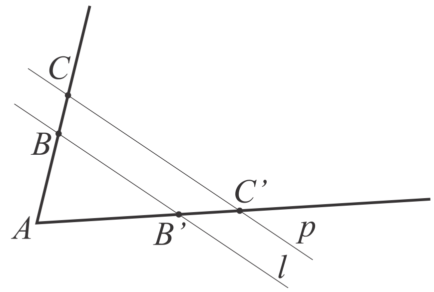
Juxtapose ΔABC and $\Delta A'B'C'$ in such a way that their congruent sides AC and $A'C'$ would coincide and the vertices B and B' would lie on the opposite sides of $A'C'$ (see the picture). Connecting vertices B and B' we will get 2 isosceles triangles, BAB' and BCB'. In the isosceles triangle angles at the base are congruent, so $\angle ABB' = \angle AB'B$, and $\angle CBB' = \angle CB'B$, therefore $\angle ABC = \angle AB'C$ and triangle ΔABC is congruent to the triangle $\Delta A'B'C'$.

Congruency tests.

- **SAS** (Side-Angle-Side): If two pairs of sides of two triangles are equal in length, and the included angles are equal in measurement, then the triangles are congruent.

- **SSS** (Side-Side-Side): If three pairs of sides of two triangles are equal in length, then the triangles are congruent.

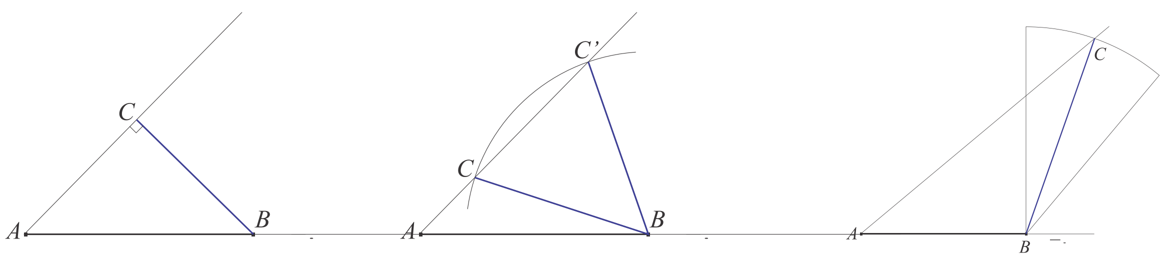
- **ASA** (Angle-Side-Angle): If two pairs of angles of two triangles are equal in measurement, and the included sides are equal in length, then the triangles are congruent.



Based in these criteria, we can see that a triangle is defined by either three sides, or by the side and two adjacent angles, or by the two sides and the angle formed by them. And what about another combination of sides and angles? Do three angles define a triangle? Are the two triangles with congruent angles are congruent? No, just see the example on the picture. Two parallel lines l and p intersect two sides of the angle $\angle A$. Two triangles $\Delta ABB'$and $\Delta ACC'$ are formed. Angle A is the common angle, angles $\angle ABB'$and $\angle ACC'$ are congruent, as well as angles $\angle AB'B$ and $\angle AC'C$ as the corresponding angles formed by transversal crossing two parallel lines. Triangles $\Delta ABB'$and $\Delta ACC'$ are not congruent.

Let's take a look on the **AAS** combination, two angles and the side, not adjacent to both angles, only to one angle. This case can easy by reduced to **ASA** criteria, since the third angle is always known.

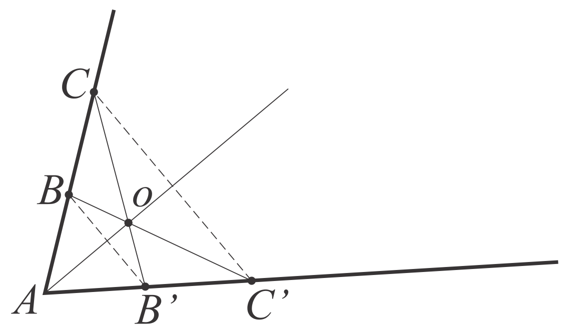
SSA (two sides and the angle not formed by these two sides) condition is more interesting, since several cases can be considered.



The first case represents the shortest possible second side and the right triangle is formed, second case represents the situation where the second side is bigger than the distance from point B to the ray AC', but smaller then the length of the segment AB. Two triangles are satisfying the condition SSA. The third case shows that if the length of the second side is equal or bigger than the length of the segment AB, the only one triangle satisfy the SSA condition.

Exercise.

1. On one side of an angle $\angle A$, the segments AB and AC are marked, and on the other side the segments $AB' = AB$ and $AC' = AC$. Prove that the lines BC' and $B'C$ met on the bisector of the angle $\angle A$.



Given:

$AB' = AB$, $AC' = AC$, $O = BC' \cap B'C$
Prove: $\angle CAO = \angle OAC'$

Statement	Reason	Conclusion
$\Delta ABC' = \Delta AB'C$	because $AC = AC'$, $AB = AB'$ and angle A is a common angle	$\angle ACB' = \angle AC'B$ $CB' = BC'$
$CO = OC'$	Because $\angle ACC' = \angle AC'C$ as angles at the base of the isosceles triangle, $\angle ACB' = \angle AC'B$ as the corresponding angles of the equal triangles. So the triangle COC' is an isosceles triangle (converse theorem, need to be proved).	$\Delta AOC = \Delta AOC'$ by the SAS test. Therefore $\angle OAC = \angle OAC'$ $\overrightarrow{AO}$ is a bisector.

2. Prove that a triangle that has two congruent angles is isosceles.