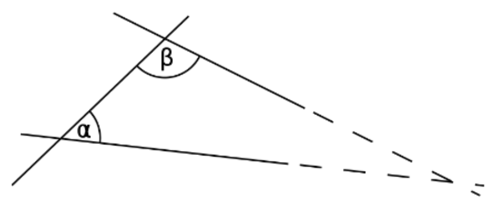
Math 5a. Classwork14

Two of the most important building blocks of geometric proofs are axioms and postulates. Axioms and postulates are essentially the same thing: mathematical truths that are accepted without proof. Their role is very similar to that of undefined terms: they lay a foundation for the study of more complicated geometry. One of the greatest Greeks (*Euclid*) achievements was setting up such rules for plane geometry. This system consisted of a collection of undefined terms like point and line, and five axioms from which all other properties could be deduced by a formal process of logic. Four of the axioms were so self-evident that it would be unthinkable to call any system a geometry unless it satisfied them:

1. A straight line may be drawn between any two points.
2. Any terminated straight line may be extended indefinitely.
3. A circle may be drawn with any given point as center and any given radius.
4. All right angles are equal.

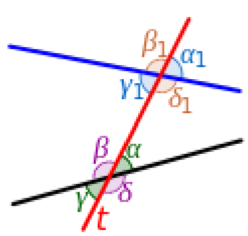
But the fifth axiom was a different sort of statement:

5. If two straight lines in a plane are met by another line, and if the sum of the internal angles on one side is less than two right angles, then the straight lines will meet if extended sufficiently on the side on which the sum of the angles is less than two right angles.

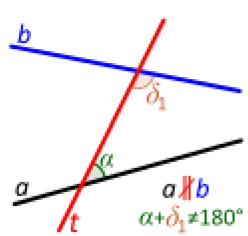


Mathematicians found alternate forms of the axiom that were easier to state, for example:

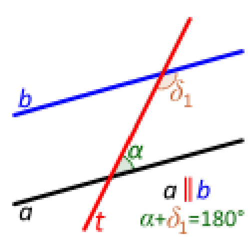
5'. For any given point, not on a given line, there is exactly one line through the point that does not meet the (or parallel to) given line.



Eight angles of a transversal. (Vertical angles such as γ and α are always congruent.)

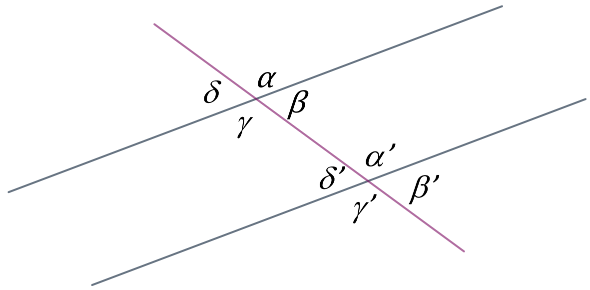


Transversal between non-parallel lines. Consecutive angles are not supplementary.



Transversal between parallel lines. Consecutive angles are supplementary.

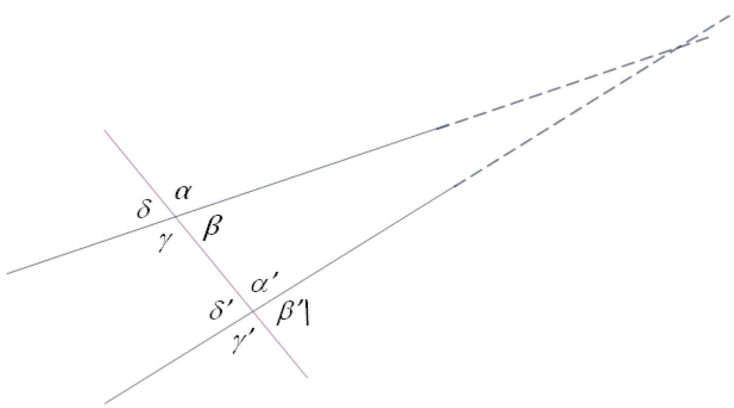
Angles formed when transversal crosses two parallel lines:



1. All angles with and without ’ are corresponding angles: α and α’, β and β’ and so on.
2. β and α’ are consecutive interior,
3. γ and δ’ are consecutive interior,
4. β’ and α are consecutive exterior,
5. γ’ and δ are consecutive exterior,
6. β and δ’ are alternate interior,
7. γ and α’ are consecutive interior,
8. α and γ’ are consecutive exterior,
9. δ and β’ are consecutive exterior,

If two consecutive interior angles formed by transversal intersecting two lines are supplementary then two lines are parallel. If two parallel lines are intersected by the third line, then the interior consecutive angles are supplementary.

On this picture $\beta + \alpha' < 180^\circ$, so lines are not parallel and will intersect at the side of angles β and α’, if extended. $\gamma + \delta' > 180^\circ$ (greater than strait angle), lines will not intersect at that side.



The Euclid’s *Elements* also include the following five "common notions":

1. Things that are equal to the same thing are also equal to one.

   If $a = c, \quad b = c, \;\; then\; a = b.$

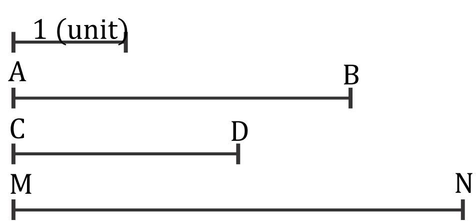
2. If equals are added to equals, then the wholes are equal (Addition property of equality).

   If $a = b,\; c = d; then\; a + c = b + d$

3. If equals are subtracted from equals, then the remainders are equal (Subtraction property of equality).
4. Things that coincide with one another are equal to one another (Reflexive Property).
5. The whole is greater than the part.

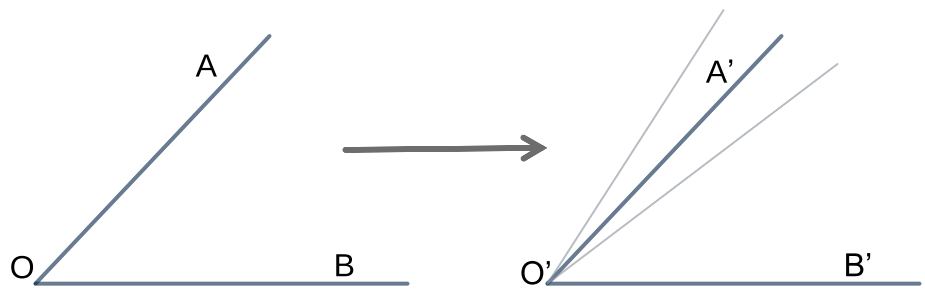
Congruent and non-congruent segments. Two segments are congruent if they can be laid on onto the other so that their endpoints coincide. Suppose that we put the segment [AB] onto the segment [CD] (pict. below) by placing the point A at the point C and aligning the ray [AB) with the ray [CD). If, as the result of this, the points B and D merge, then the segments [AB] and [CD] are congruent, or equal. Otherwise, they are not congruent, and the one which makes a part of the other is considered smaller.

We can introduce the concept of sum of several segments, we can subtract one segment from another. On the picture below, each segment contains several unit segments. Using compass find the length of each segment (in the unit segments).



How to construct the segment equal to another segment?

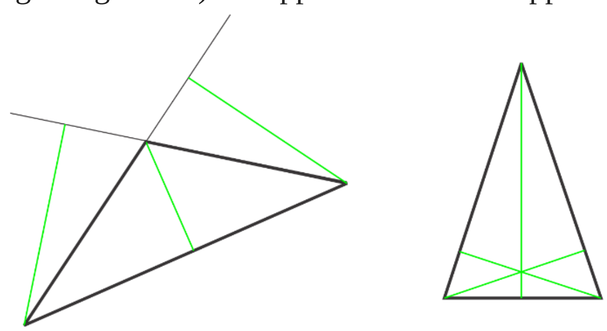
In the very similar way, we can define when two angles are congruent. Two angles are congruent if by moving one of them it is possible to superpose it with the other. The point O should be superposed with point O’, ray OB is coincided with ray O’B’. If the ray OA will superpose with ray O’A’ then 2 angles are congruent.



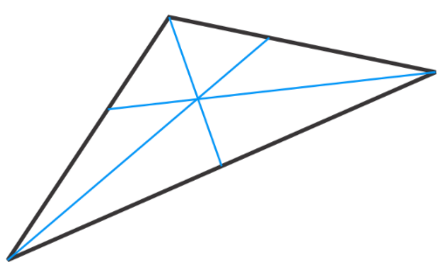
Special segments of a triangle.

From each vertex of a tringle to the opposite side 3 special segment can be constructed.

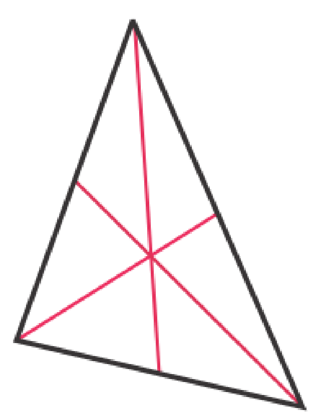
An **altitude** of a triangle is a straight line through a vertex and perpendicular to (i.e. forming a right angle with) the opposite side. This opposite side is called the *base* of the altitude, and the point where the altitude intersects the base (or its extension) is called the *foot* of the altitude.



An **angle bisector** of a triangle is a straight line through a vertex which cuts the corresponding angle in half.



A **median** of a triangle is a straight line through a vertex and the midpoint of the opposite side, and divides the triangle into two equal areas.



In elementary geometry the word *congruent* is often used as follows. The word *equal* is often used in place of *congruent* for these objects.

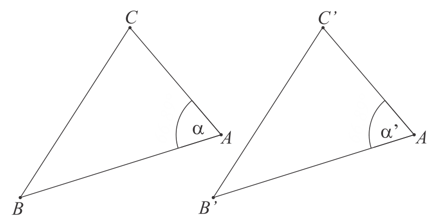
- Two line segments are congruent if they have the same length.
- Two angles are congruent if they have the same measure.
- Two circles are congruent if they have the same diameter.

Sufficient evidence for congruence between two triangles can be shown through the following comparisons:

- **SAS** (Side-Angle-Side): If two pairs of sides of two triangles are equal in length, and the included angles are equal in measurement, then the triangles are congruent.

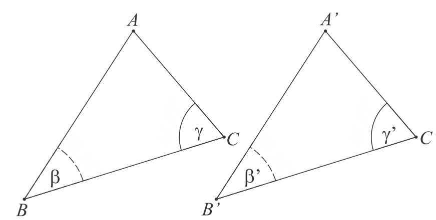
- **SSS** (Side-Side-Side): If three pairs of sides of two triangles are equal in length, then the triangles are congruent.

- **ASA** (Angle-Side-Angle): If two pairs of angles of two triangles are equal in measurement, and the included sides are equal in length, then the triangles are congruent.



SAS (Side-Angle-Side).
ABC and A'B'C' are two triangles such that

$AC = A'C', AB = A'B'$, and $\angle A = \angle A'$ We need to prove that these triangles are congruent. Superimpose ΔABC onto $\Delta A'B'C'$ in such a way that vertex A would coincide with A', the side AC would go along $A'C'$, and side AB would lie on the same side of $A'C'$ as $A'B'$. Since AC is congruent to $A'C'$, the point C will merge with point C'., due to the congruence of $\angle A$ and $\angle A'$, the side AB will go along $A'B'$ and due to the congruence of these sides, the point B will coincide with B'. Therefor the side BC will coincide with $B'C'$.



ASA (Angle-Side-Angle) ABC and $A'B'C'$ are two triangles, such that

$\angle C = \angle C'$, $\angle B = \angle B'$, and $BC = B'B'$.

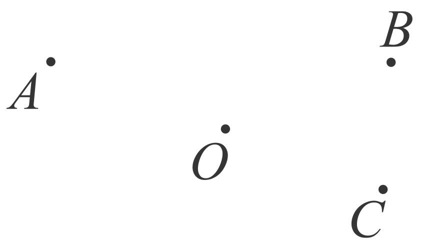
We need to prove, that these triangles are congruent. Superimpose the triangles in such a way that point C will coincide with point C', the side CB would go along $C'B'$ and the vertex A would lie on the same side of $C'B'$ as A'. Then: since CB is congruent to $C'B'$, the point B will merge with B', and due to congruence of the angle $\angle B$ an $\angle B'$, and $\angle C$ and $\angle C'$, the side BA will go along B'A', and side CA will go along C'A'. Since two lines can intersect only at 1 point, the vertex A will have merge with A'. Thus, the triangles are identified and are congruent.

Exercises:

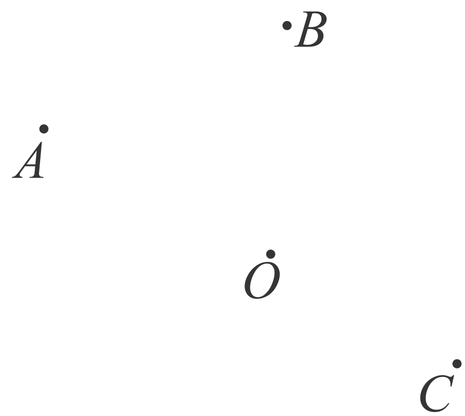
1. Prove that bisectors of two supplementary angles are perpendicular to each other (draw a sketch, use ruler and protractor).

2. Ratio of the measures of two supplementary angles is 2:3. What are the measures of the angles?

3. Do the points A, O, and C lie on the same line?

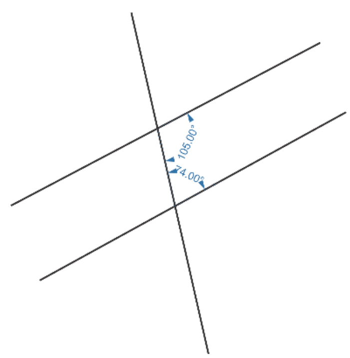


   a. Angle $\angle AOB = 137°$
      Angle $\angle BOC = 43°$



   b. Angle $\angle AOB = 65°$
      Angle $\angle BOC = 116°$

4. Are these two lines parallel?



5. Proof that the sum of the angles of a triangle is 180 degrees.