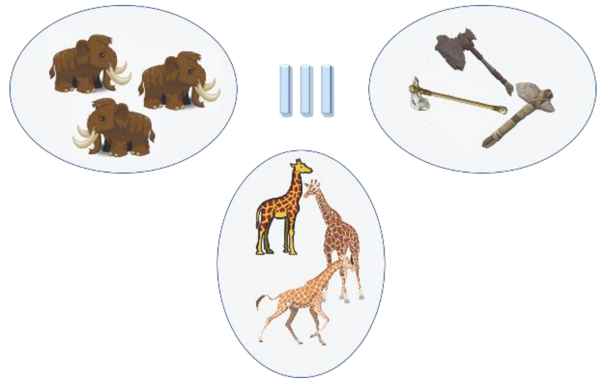
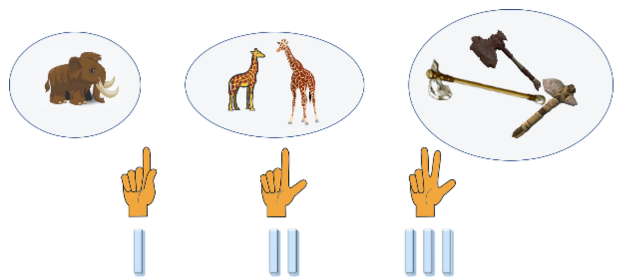
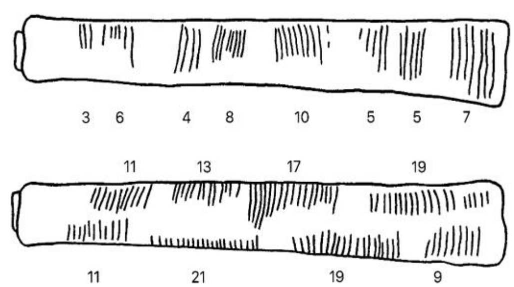
Math 4a. Classwork 2.

Very long time ago, our ancestors realized that each group of objects possesses a quantitative property: how many objects are there in the group? This property doesn't depend on the nature of the objects themselves. Groups can be compared based on this property, determining which group has more and which group has fewer items. This gave rise to the concept of "number".



Initially, prehistoric people compared the number of objects in the group with the count of fingers– since hands were ever-present! Then they began scratching marks on wood and bones as a means to record quantities. One of the oldest known examples of such bones is the *Ishango bone*, which dates back to around 20,000-25,000 years ago.





Historians continue to debate the purpose of the *Ishango bone*. Some think that might have been an early calculating tool due to its tally marks grouped in a specific manner. (Of course, we cannot possibly know how exactly it was used.)

The next step in the progress of math is the creation of the arithmetic operations. What can we do with numbers? Numbers can be added together. At first, the operation +1 was developed:

$$||| + | = ||||$$

(There were no “+” sign, but we can use it for convenience.)

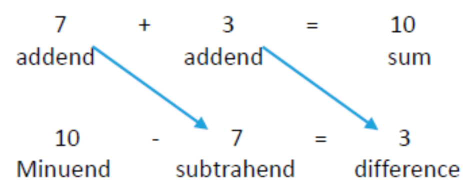
Then addition of two groups made the big progress:

$$||||||| + ||| = ||||||||||$$

All the numbers we now use to count are called natural numbers. In the later times, various systems of writing numbers were developed, and in our present decimal system we can write the same as:

$$7 + 3 = 10$$

The operation of subtraction for natural numbers is the way to find the number that, when added to the numbers we are subtracting, results in the initial number. There are special names for the numbers in this operation:



$$addend + addend = sum$$

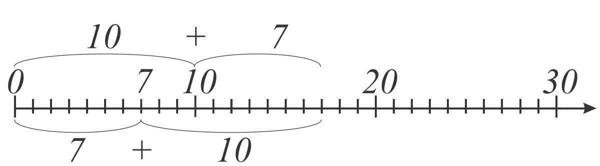
$$minuend - subtrahend = difference$$

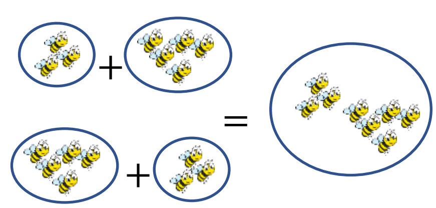
So, if we add *subtrahend* to a *difference*, the result should be the *minuend*.

The operation of addition has a few properties:

- it’s commutative: it doesn’t matter what addend goes first, the sum will not change.
- It’s associative: if three terms are added together, it doesn't matter how they are added—whether the first two and then the third or the second plus the third and then the first—the result will remain the same.

Commutative and associative properties of addition are intuitively easy to understand.

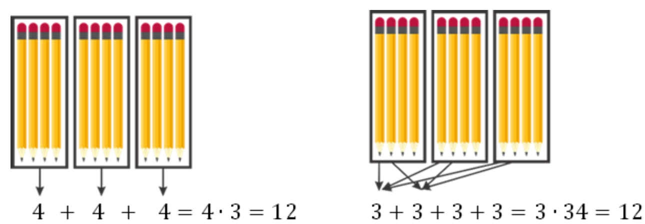




After that, multiplication was introduced, as addition of the same addend (term) several times.

$$c \cdot b = \underbrace{c + c + c + \cdots + c}_{b\ times} = \underbrace{b + b + b + \cdots + b}_{c\ times} = a$$

The result of multiplication is called *product*, and the participants of the operation are called *factors*. c and b are factors, and a is a product.



How many pencils are there in three boxes, if there are four pencils in each box?

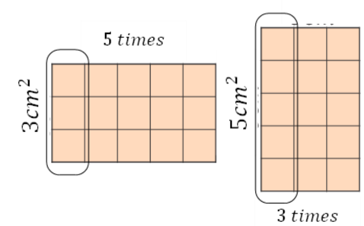
$3 \cdot 4 = 12$; three and four are factors, 12 is a product.

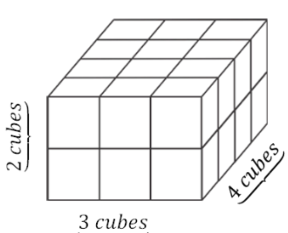
Multiplication also has properties:

- it's commutative. It doesn't matter what factor goes first, the product will not change.
- It's associative. If three factors are multiplied, it doesn't matter whether we first get the product of the first two and then multiply by the third factor or the second and third multiplied first, and then the result is multiplied by the first factor—the product will still be the same.

The commutative property of multiplication can be also illustrated by calculating of the area of a rectangle:

$$S = 3cm^2 \times 5times = 5cm^2 \times 3times = 15cm^2$$





The associative property can be shown by calculating of the volume of parallelepiped. How many cubed are stacked together into the parallelepiped?

First, we can multiply four by three to find out how many cubes are in the horizontal slice, and then multiply by 2 (the number of slices). Or multiplication of three by two will provide the number of cubes on the front slice, and there are four such slices.

$$(3 \cdot 4) \cdot 2 = (2 \cdot 4) \cdot 3 = (2 \cdot 3) \cdot 4 = 2 \cdot 3 \cdot 4 = 24$$

	Commutative	Associative	Identity
Addition	$a + b = b + a$	$(a + b) + c = a + (b + c)$	$a + 0 = a$
Multiplication	$a \cdot b = b \cdot a$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$	$a \cdot 1 = a$
Distributive property of multiplication over addition	$a \cdot (b + c) = a \cdot b + a \cdot c$		

Distributive property can be illustrated with the following problem:

The farmer put green and red grapes into boxes. Each box contains 5lb of grapes. How many pounds of green and red grapes altogether did the farmer put into boxes if he had 10 boxes of green and 8 boxes of red grapes?

We can first find out how many boxes of grapes the farmer has altogether and multiply it by 5lb in each box, or we can find out the weight of white and red grapes in the boxes and then add it.

$$5 \times (10 + 8) \quad or \quad 5 \times 10 + 5 \times 8$$

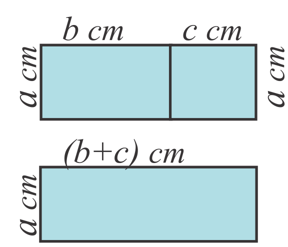
$$5 \times 18 = 50 + 40; \qquad 90 = 90$$

Another example:

The combined area of these two rectangles is the sum of two areas, $S = a \times b + a \times c$

The rectangle with one side a cm and the other side $(b + c)$ cm will have exactly the same area, $S = a \times b + a \times c = a \times (b + c)$. (I have used the variables a, b, and c instead of numbers to show the distribution property in a general way).

Using the distributive property, we can factor out the common factor of two terms of the expression, for example:



$$6 \cdot 7 + 6 \cdot 3 = 6(7 + 3) = 6 \cdot 10 = 60$$

Properties of the arithmetic operations can be used to simplify the calculations.

Examples:

We need to multiply a two-digit number by a single digit number.

$$28 \cdot 9 = (20 + 8) \cdot 9 = 20 \cdot 9 + 8 \cdot 9 = 180 + 72$$

$$180 + 72 = 100 + 80 + 70 + 2 = 100 + 150 + 2 = 252$$

Another example: how to find the value of the expression: $250 \cdot 61 - 25 \cdot 390$

Of cause, it can be calculated in a very straightforward way:

$$250 \cdot 61 - 25 \cdot 390 = 15250 - 9750 = 5500$$

Or we can use the properties of multiplication (which property has been used here?):

$$25 \cdot 390 = 25 \cdot 39 \cdot 10 = 250 \cdot 39$$

$$250 \cdot 61 - 25 \cdot 390 = 250 \cdot 61 - 250 \cdot 39 = 250(61 - 39) = 250 \cdot 22$$

$$= 25 \cdot 22 \cdot 10 = (20 + 2) \cdot 25 \cdot 10 = (20 \cdot 25 + 2 \cdot 25) \cdot 10 = 550 \cdot 10 = 5500;$$

Exercises:

1. Do the calculations in your head:

   $25 \cdot 8;$ $\quad 132 + 221;$ $\quad 248 - 134;$ $\quad 9 \cdot 38;$ $\quad 321 \cdot 41 + 59 \cdot 321;$

2. Factorize (represent as a product of two or more factors):

   Example: $35 = 3 \cdot 7;$ $\quad 100 = 4 \cdot 5 \cdot 5;$

   $$21, \quad 24, \quad 30, \quad 49, \quad 75, \quad 1000$$

3. Evaluate (what is the best way to compute it? Hint: use commutative property):

   $a.$ $72 + 59 + 97 + 28 + 41;$ $\quad b.$ $32 + 34 + 36 + 38;$

   $c.$ $5 \cdot 19 \cdot 5 \cdot 3 \cdot 2 \cdot 2;$ $\quad d.$ $715 - 99 - 101;$

   $e.$ $(629 + 56) - 629;$ $\quad e.$ $232 - (95 + 132);$

4. Evaluate (what is the best way to compute it? Hint: use the distributive and/or commutative properties):

   $a.$ $23 \times 15 + 15 \times 77;$ $\quad b.$ $79 \times 21 - 69 \times 21;$ $\quad c.$ $340 \times 7 + 16 \times 70;$

   $d.$ $250 \times 61 - 25 \times 390;$ $\quad e.$ $67 \times 58 + 33 \times 58;$ $\quad f.$ $55 \times 682 - 45 \times 682$

   $g.$ $5 \cdot 4 + 5 \cdot 44 + 5 \cdot 444 + 5 \cdot 4444;$

   $h.$ $26 \cdot 25 - 25 \cdot 24 + 24 \cdot 23 - 23 \cdot 22 + 22 \cdot 21 - 21 \cdot 20 + 20 \cdot 19 - 19 \cdot 18 + 18 \cdot 17 - 17 \cdot 16 + 16 \cdot 15 - 15 \cdot 14;$