Math 7: Handout 28
Trigonometry 8: Inverse trigonometric functions: arc-sine and arc-cosine [2023/05/14]

Previously, we solved trigonometric equations of type $\sin\alpha = \sin c$ and obtained an infinite number of solutions, different by a whole number of full-circles (2π). Even within one full circle ($0 \le \alpha < 2\pi$), there may be two solutions because the horizontal line $y = \sin c$ may intersect the trigonometric circle at two points.

Arc-sine function

Now, how does one solve a general equation $\sin\alpha = A$? We know that there are infinitely many solutions different by ($2\pi n$), so we need to know only one to find all the others. Suppose we find one angle α_0 such that $\sin\alpha_0 = A$; then, all other solutions (see the figure, left) can be written as

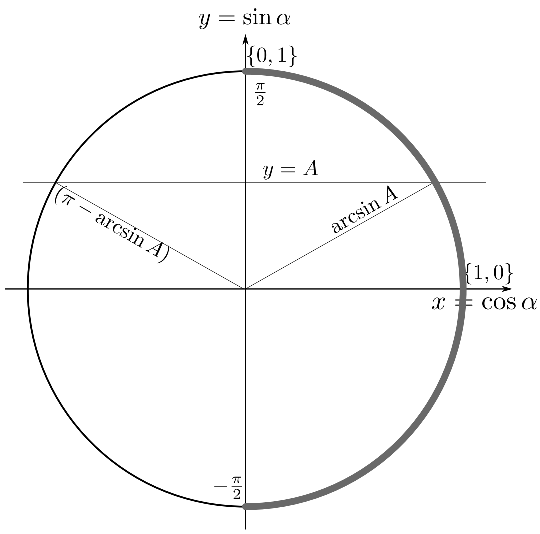
$$\left[\begin{array}{ll} \alpha & = 2\pi n + \alpha_0\,, \\ \alpha & = 2\pi n + \pi - \alpha_0\,, \end{array}\right. \quad \forall n = 0, \pm1, \pm2, \ldots \in \mathbb{Z}$$

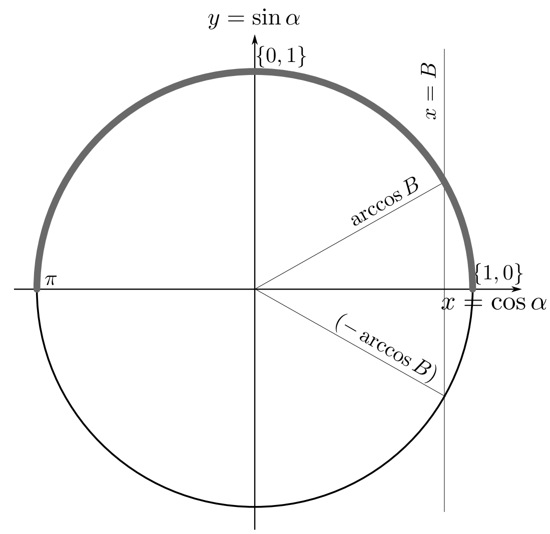
While it does not matter which α_0 we choose (from infinitely many), it is conventional to choose it to be in the range $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, so that there is *only single value* α_0 corresponding to any $-1 \le A \le 1$. This is called arc-sine function,

$$\alpha_0 = \arcsin A\,, \quad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}\,.$$

All possible angles that can be produced by $\arcsin$ are shown on the trigonometric circle (left). The value of $\arcsin A$ can be looked up in a table, but it is, in general, an infinite decimal fraction (like π; such numbers are called transcendental). It is conventional to leave the $\arcsin A$ as-is in the answer:

$$\sin\alpha = A \quad\Longleftrightarrow\quad \left[\begin{array}{ll} \alpha & = 2\pi n + \arcsin A\,, \\ \alpha & = 2\pi n + \pi - \arcsin A\,, \end{array}\right. \qquad \forall n \in \mathbb{Z}$$





Arc-cosine function

Now, what about the equation $\cos\beta = \cos d = B$? You should remember that $\cos\beta = \cos(-\beta)$, so for every solution, its negative is also a solution, so

$$\left[\begin{array}{ll} \beta & = 2\pi n + \beta_0\,, \\ \beta & = 2\pi n - \beta_0\,, \end{array}\right. \quad \forall n = 0, \pm1, \pm2, \ldots \in \mathbb{Z}$$

(the ($2\pi n$) piece includes negative and positive cases). It is conventional to choose the angle $0 \le \beta_0 < \pi$, so that it is *unique* for any $-1 \le B \le 1$. All possible angles β_0 are shown in the figure (right), so that

$$\cos\beta = b \quad\Longleftrightarrow\quad \beta = 2\pi n \pm \arccos B\,, \quad \forall n = 0, \pm1, \pm2, \ldots \in \mathbb{Z}$$

The values of $\arcsin A$ and $\arccos A$ for some standard values of A are summarized in the table

A	-1	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\arcsin A$	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\arccos A$	π	$5\pi/6$	$3\pi/4$	$2\pi/3$	$\pi/2$	$\pi/3$	$\pi/4$	$\pi/6$	0

HOMEWORK

1. Solve the following equations:
  (a) $\sin x = \frac{2}{3}$;
  (b) $\cos x = -\frac{3}{4}$;
  (c) $\cos(x - \frac{\pi}{2}) = \frac{1}{4}$;
  (d) $\sin(\frac{\pi}{2} + x) = \frac{1}{3}$.

2. Solve the following equations:
  (a) $12(\cos x)^2 - \cos x - 1 = 0$;
  (b) $(\sin x)^2 + 3(\cos x)^2 = 2$;

3. Solve inequalities
  (a) $\cos x \leq \frac{1}{3}$;
  (b) $\sin x > \frac{1}{4}$.

4. Let's assume that for some angle α, $\sin \alpha = A$. Can you find the sum $(\arcsin A + \arccos A)$?
Recall that swapping the axes $x \leftrightarrow y$ is equivalent to swapping $\cos \alpha \leftrightarrow \sin \alpha$.