Math 7: Handout 27
Trigonometry 7: Trigonometric identities; reduction formulas [2023/05/07]

Reduction formulas

We already know that $\sin\alpha$ and $\cos\alpha$ are periodic with period 2π (do not change if you add 2π to α) and change sign if you add π to α. Since $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$, it does not change if you add π to α ($\tan\alpha$ is periodic with period π). What if one adds a fraction of angle to α, i.e. $\alpha \to (\alpha + \frac{\pi}{2})$? On a trigonometric circle (see the figure), this is equivalent to a turn by $90°$ counter-clockwise, and

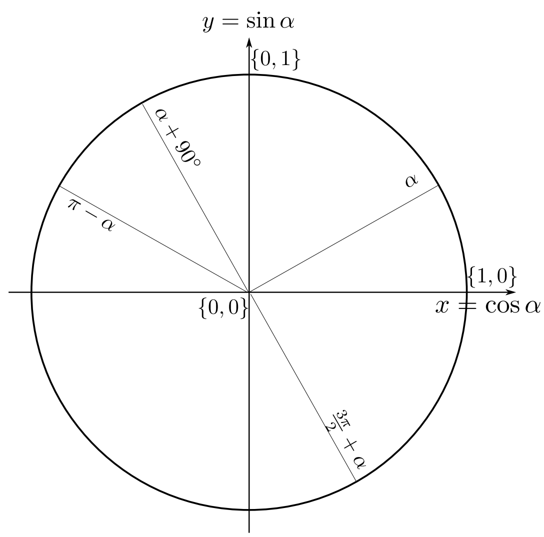
$$\begin{aligned}\sin(\alpha + 90°) &= \quad \cos\alpha\,,\\ \cos(\alpha + 90°) &= -\sin\alpha\,,\end{aligned}$$

It is evident that $\sin$ changes to $\cos$, and $\cos$ to $\sin$ for any angle α because x and y axes trade places. However, the sign can be confusing. Thankfully, there is a very simple rule of thumb how to know the sign of the result:

Assume for a moment that angle α is acute ($0 < \alpha < 90°$) and see in which quadrant the argument ($\alpha + 90°$) ends up!

For the above example, it is quadrant II, where $\sin(\alpha + 90°)$ is positive; therefore, the right-hand side ($\cos\alpha$) must also be positive, so the sign is "$(+)$". On the other hand, $\cos(\alpha + 90°)$ is negative, so the sign is "$(-)$" in the second line.

	I $0 < \alpha < \frac{\pi}{2}$	II $\frac{\pi}{2} < \alpha < \pi$	III $\pi < \alpha < \frac{3\pi}{2}$	IV $\frac{3\pi}{2} < \alpha < 2\pi$
$\sin x$	$+$	$+$	$-$	$-$
$\cos x$	$+$	$-$	$-$	$+$
$\tan x$	$+$	$-$	$+$	$-$
$\cot x$	$+$	$-$	$+$	$-$



Let's look at another example and use radians instead of angles; what if the argument is $(\pi - x)$? In this case, the trigonometric circle is "reflected" about the y-axis, so the axes x and y themselves are not swapped; however, the sign of x coordinate changes to the opposite, therefore

$$\begin{aligned}\sin(\pi - \alpha) &= \quad \sin\alpha\,,\\ \cos(\pi - \alpha) &= -\cos\alpha\,,\end{aligned}$$

The same rules apply to tangent and cotangent, and we can now summarize the rules for reduction as follows:

1. If the angle is changed by an odd number of $\frac{\pi}{2}$ (that is, $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, etc), change to co-function ($\sin \to \cos$, $\cos \to \sin$, $\tan \to \cot$, $\cot \to \tan$) ; if the angle is changed by an even number of $\frac{\pi}{2}$ (that is $\pm\pi$, $\pm 2\pi$, etc) the function is not changed.
2. if the initial function is positive in the quadrant for small (acute) angle α, keep the sign the same; if it is negative, change it to the opposite.

As the another example, let's find the value of $\tan(\frac{3\pi}{2}+\alpha)$. First, we know that the function will change from $\tan$ to $\cot$; second, the angle $(\frac{3\pi}{2}-\alpha)$ is in the 4th quadrant (assuming that the angle α is acute), where $\tan$ is negative; therefore, the reduction rules give

$$\tan(\frac{3\pi}{2}+\alpha) = -\cot\alpha.$$

Simple trigonometric identities

We already know that $(\sin\alpha)^2 + (\cos\alpha)^2 = 1$. This identity helps us find $\cos\alpha$ and $\tan\alpha = \frac{1}{\cot\alpha}$ if we know that, for example $\sin\alpha = c$, but up to a sign:

$$\cos\alpha = \sqrt{1-(\sin\alpha)^2} = \sqrt{1-c^2}\,,$$

$$\tan\alpha = \frac{1}{\cot\alpha} = \frac{\sin\alpha}{\cos\alpha} = \pm\frac{c}{\sqrt{1-c^2}}$$

To know the sign, we also need to know which quadrant α belongs to.

For example, if $\sin\alpha = -\frac{3}{5}$, and α is in the 4th quadrant, then $\cos\alpha > 0$ and $\tan\alpha < 0$. Therefore,

$$\cos\alpha = \sqrt{1-(3/5)^2} = \frac{4}{5}\,,$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-3/5}{4/5} = -\frac{3}{4}\,.$$

The other useful identities are for tangent and cotangent,

$$1 + (\tan\alpha)^2 = \frac{1}{(\cos\alpha)^2}\,,$$

$$1 + (\cot\alpha)^2 = \frac{1}{(\sin\alpha)^2}\,,$$

If we know the tangent of some angle $\tan\alpha = d$, it is also easy to find $\cos\alpha$ and $\sin\alpha$:

$$\cos\alpha = \pm\frac{1}{\sqrt{1+(\tan\alpha)^2}} = \pm\frac{1}{\sqrt{1+d^2}}\,,$$

$$\sin\alpha = \tan\alpha\cdot\cos\alpha = \pm\frac{d}{\sqrt{1+d^2}}\,.$$

and the signs depend on which quadrant α is in.

Homework

1. Reduce the angle argument to α in the following expressions:

(a) $\cos(x-\frac{\pi}{2})$;
(b) $\sin(\pi+x)$;
(c) $\tan(2\pi-x)$;
(d) $\cos(x+\frac{3\pi}{2})$;
(e) $\sin(\frac{3\pi}{2}-x)$;
(f) $\cot(x+\frac{3\pi}{2})$.

2. If you know that $\sin x = \frac{5}{13}$ and $0 < x < \frac{\pi}{2}$, find the values
(a) $\cos(x+\frac{\pi}{2})$;
(b) $\cot(\frac{3\pi}{2}-x)$;
(c) $\sin(2\pi-x)$?

3. If you know that $\tan x = -\frac{1}{2}$ and $\frac{\pi}{2} < x < \pi$, can you find
(a) $\sin(x-\frac{\pi}{2})$;
(b) $\tan(\pi-x)$;
(c) $\sin(x+\frac{3\pi}{2})$?

4. Show that $\sin x + \sin(x+\frac{\pi}{3}) + \sin(x+\frac{2\pi}{3}) + \sin(x+\pi) + \sin(x+\frac{4\pi}{3}) + \sin(x+\frac{5\pi}{3}) = 0$.