Math 7: Handout 23
Trigonometry 3: The Trigonometric Circle [2023/04/02]

Radians

Until now, we have been measuring angles in degrees, which are defined by saying that a full turn corresponds to $360°$.

An alternative way to measure angles is by radians, which are defined in the following way: given an angle α, it's measure in radians is the ratio of an arc of circumference with angle α by the radius of the circumference.

For example, the angle $360°$ corresponds to a full circle. Since the perimeter of a circle is $2\pi R$, dividing by R gives:

$$360° \leftrightarrow 2\pi \text{ rad.}$$

In the same way, half a circle corresponds to an angle of π radians. By similar arguments, we can translate all the angles that appeared in our previous table:

Trigonometric Functions							
Function	Notation	Definition	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sine	$\sin(\alpha)$	$\frac{\text{opposite side}}{\text{hypotenuse}}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cosine	$\cos(\alpha)$	$\frac{\text{adjacent side}}{\text{hypotenuse}}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tangent	$\tan(\alpha)$	$\frac{\text{opposite side}}{\text{adjacent side}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

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Trigonometric Circle

A very useful tool in understanding the trigonometric functions is the *trigonometric circle* (see figure below): in order to find the sine and cosine of a positive angle α, we just have to "walk" around the circle a distance α, starting from the point $(1, 0)$ in anti clockwise direction. Then the coordinates of the point we arrive at are $(\cos\alpha, \sin\alpha)$. For α negative, we define the sine and cosine in the same way, but walking in the clockwise direction.

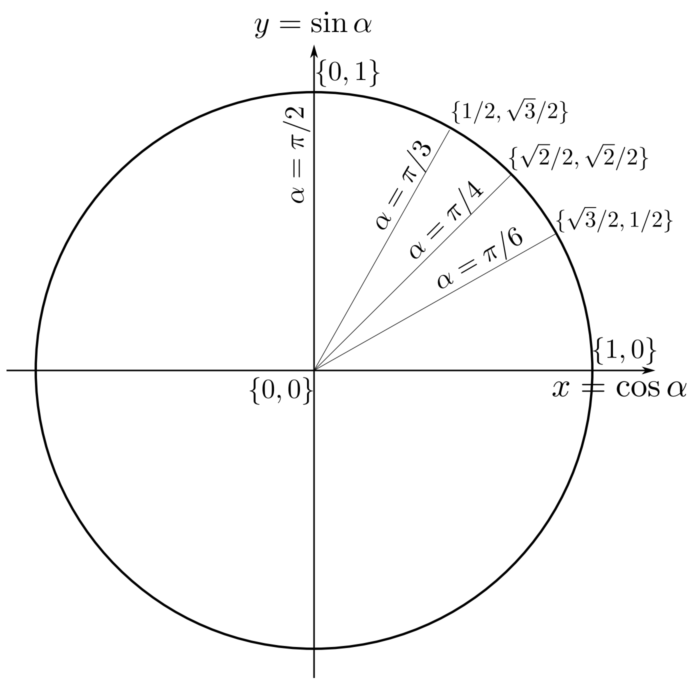

FIGURE 1. Trigonometric circle: in order to find the sine and cosine of angle α, we just have to "walk" around the circle a distance α, starting from the point $(1, 0)$. Then the coordinates of the point we arrive at are $(\cos\alpha, \sin\alpha)$.

By looking at the values of sine and cosine as we go around the trigonometric circle, it becomes self-evident that

- $\sin(x)$ is periodic and repeats itself after 2π: $\sin(x + 2\pi) = \sin x$;
- $\sin(x)$ changes its sign every time x changes by π: $\sin(x + \pi) = -\sin x$;
- the above is also true for the cosine: $\cos(x + \pi) = -\cos x$, $\cos(x + 2\pi) = \cos x$;
- over the period 2π, both sine and cosine cross zero value two times: $\sin 0 = \sin\pi = 0 = \cos\frac{\pi}{2} = \cos\frac{3\pi}{2}$;
- $\sin x$ reaches it's maximum value 1 at $x = \frac{\pi}{2}$, decreases from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, reaches its minimum value -1 at $x = \frac{3\pi}{2}$, increases to $\frac{5\pi}{2}$ and then repeats itself;
- $\cos x$ decreases from $(+1)$ to (-1) for $0 < x < \pi$ and increases back to $(+1)$ for $\pi < x < 2\pi$.

HOMEWORK

1. Draw a large trigonometric circle. Then, remembering that 2π corresponds to a full circle, find the points corresponding to (write the corresponding letter on the correct point)

(a) π
(b) $\frac{3\pi}{2}$
(c) $\frac{3\pi}{4}$
(d) $-\frac{5\pi}{4}$
(e) 11π
(f) -3π
(g) $\frac{25\pi}{3}$
(h) $-\frac{19\pi}{6}$

2. Now use your trigonometric circle and figure 1 to complete this table:

Point	Sine	Cosine
(a)	0	-1
(b)		
(c)		
(d)		
(e)		
(f)		
(g)		
(h)		

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3. Using the trigonometric circle, check where appropriate:

x	$\sin x \geq \sqrt{3}/2$	$1/2 < \sin x < \sqrt{3}/2$	$-\sqrt{2}/2 < \sin x \leq 1/2$	$\sin x \leq -\sqrt{2}/2$
$\pi/7$			✓	
$2\pi/7$				
$-3\pi/5$				
$5\pi/8$				
$25\pi/9$				

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4. Using the trigonometric circle, show that $\cos x = \sin(x + \pi/2)$ for any angle x.

5. Find all real numbers x such that $(\sin x)^2 = 3/4$