MATH 6: HANDOUT XXVI
BONUS TOPICS: PARITY AND GAMES OF STRATEGY

PARITY

PARITY is the property of a number being either even or odd. For example, it is easier to say "The parity of the number of upside-down cups does not change" than "If the number of upside-down cups is even, it stays even; if this number is odd, it remains odd."

GAMES OF STRATEGY

In games of strategy, we need to look for a winning strategy: a set of instructions that one of the players can follow in order to ensure his or her victory. First, these instructions should be PRECISE and COMPLETE: whatever the opponents' move is, the player should be able to reply with a move according to the instructions. Second, the winning strategy should be verifiable: it should be possible to explain why this set of rules is certain to bring victory to the player.

EXERCISES

1. Savir the Junior Hacker reprogrammed the elevator in the 100-story Boogle Corporation building: only two buttons are currently working. The first button sends the elevator 8 floors up, and the second one 6 floors down. (The elevator will not move if it is asked to go above the 100th floor or below the 1st floor.)
   (a) The company's CEO is currently drinking coffee on the first floor. (There is no lobby floor in the building). Can he take the elevator to the 95th floor? If so, show how. If not, explain why.
   (b) Can he take the elevator to the 96th floor? If so, show how. If not, explain why.

2. The integers from 1 to 22 are written on the board in a row. Can you insert plus and minus signs between them in such a way as to get an expression that is equal to 0?

3. In the Land of Not So Far Away there live 9 happy and 9 unhappy princesses. Shmerlin the Magician has just learned three new spells. The first spell makes any two unhappy princesses of his choice happy. His second spell transforms any pair of happy princesses into unhappy ones. The third spell switches the moods of a happy princess and an unhappy one: the happy princess becomes unhappy, and the unhappy one becomes happy. Shmerlin would like to make all the princesses happy. Prove that these three spells are not sufficient for his plan to come true by following these four steps:
   (a) Explain what effect each of Shmerlin's spell has on the number of unhappy princesses.
   (b) Currently, the number of unhappy princesses is odd. Suppose that Schmerlin utters one of his spells. Prove that the number of unhappy princesses will remain odd.
   (c) Suppose that Schmerlin performs several spells in a row. Prove that the number of unhappy princesses will remain odd.
   (d) Is it possible for the number of unhappy princesses to eventually go down to zero?

4. Jack and Jelly are on the ship en route to discover Atlantis and decide to play a game. The distance between their starting point and the city of Atlantis is 15 kilometers. They take turns manning the ship and each of them can steer the ship for 1, 2, or 4 kilometers in one turn. This should never exceed the remaining distance. The captain who is in charge of the ship when they react Atlantis wins the game. If Jelly starts as the captain in charge of the ship, find who wins the game, i.e., who will be in charge of the ship when they reach Atlantis given both Jack and Jelly play optimally.

5. Terrence and Clarence are playing a game. They have 2 piles of candy, 23 pieces in each. One every turn, a player is allowed to take up to 5 candies from any single pile. (Skipping turns is not allowed.) The winner is the person who takes the last candy. Terrence goes first. Who has a winning strategy, and what is it?

6. Kevin and Deia are playing a game. They have 2 jars of cookies, one with 31 cookies and the other with 27. On every turn, a player is allowed to take up to 6 cookies from the same jar (a player has to take at least 1 cookie per turn). The person who takes the last cookie wins the game. Kevin goest first. Who has a winning strategy, and what is it?

7. Aishwarya and Valentina are playing a game. They have a pile of 56 marbles. On a single turn, a player can take up to 8 marbles from the pile. The player who cannot make his move loses the game. Aishwarya goes first. Who has a winning strategy, and what is it?

8. Sophie and Michelle take turns placing rooks on the squares of the 8×8 chessboard in such a way that the rooks cannot attack one another (in this game, the color of the rooks is irrelevant.) The player who cannot place a rook loses the game. Sophie takes the first turn. Who has a winning strategy, and what is it?