MATH 10
ASSIGNMENT 16: EULER'S FORMULA

FEB 5, 2023

Euler's formula

Recall the series from last time:

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We have discussed that it has the following properties:

1. It converges for any $x \in \mathbb{R}$
2. $E(x)E(y) = E(x+y)$
3. $E(0) = 1$
4. For small values of x, $E(x) \approx 1 + x$

Thus, if we denote

$$e = E(1) = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718281828\dots$$

then one can show that $E(x) = e^x$ (for integer x, it easily follows from above properties. For other values, this is taken as definition of e^x.)

We can also consider $E(x)$ for complex values of x. In particular, if $x = it$, t is real, then we can show that $|E(it)| = 1$, so $E(it)$ is on the unit circle (see problem 2 below). Moreover, we have the following formula:

Theorem (Euler's formula)**.** *If t is real, then*

$$E(it) = e^{it} = \cos t + i \sin t$$

In particular, $e^{i\pi} = -1$.

Partial proof of this is given in problem 2 below.

Homework

1. For what values of x do the series below converge? [You can use results discussed last time — in particular, the comparison test and the ratio test.]
   (a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$
   (b) $1 + \frac{x^2}{4} + \frac{x^6}{9} + \dots + \frac{x^{2n}}{n^2} + \dots$
   (c) $\sum \frac{(-1)^n x^n}{n(n+1)}$

2. Without using Euler formula, prove the following:
   (a) For any complex z, $\overline{E(z)} = E(\overline{z})$.
   (b) For real t, $\overline{E(it)} \cdot E(it) = 1$, so $|E(it)| = 1$.
   (c) Let $\varphi(t) = \arg(E(it))$, where arg is the argument (angle) of a complex number. Prove that $\varphi(0) = 0$, $\varphi(t_1 + t_2) = \varphi(t_1) + \varphi(t_2)$.
   (d) From the above, can you prove that $\varphi(t) = t$? This would show that $E(it)$ is a complex number with magnitude 1 and argument t, i.e.

$$E(it) = \cos(t) + i \sin(t)$$

   which is a psecial case of Euler's formula.

3. Separating in Euler's formula real and imaginary parts, show that $\sin(x)$, $\cos(x)$ can be written as series of the form $\sum a_n x^n$.

4. Use Euler's formula and identity $E(x)E(y) = E(x+y)$ to get formulas for $\sin(x+y)$, $\cos(x+y)$ in terms of $\sin(x)$, $\sin(y)$, $\cos(x)$, $\cos(y)$.

5. (a) It is known that the function $f(x) = \frac{1}{\cos x}$ can be written as a series $f(x) = 1 + a_2x^2 + a_4x^4 + \dots$. Using the formula for $\cos(x)$ from Problem 3, can you find a_2, a_4?

(b) Show that $\tan(x) = x + c_3x^3 + c_5x^5 + \dots$. Find c_3, c_5

6. A 100 men came to a party, each wearing a hat. They all left the hats in the coatroom. When leaving, they picked hats at random. What is the probability that none of them got his own hat?

This problem is in fact related to the series $E(x)$. To see that, you can split it in a sequence of steps.

Let A be the set of all possible choices of hats by people, so that $|A| = 100!$. Let $A_i \subset A$ be those choices where person i got his hat (with no restrictions on what happened with others).

Use inclusion-exclusion formula

$$\begin{aligned} |A_1 \cup A_2 \cup \dots \cup A_n| =& |A_1| + \dots + |A_n| \\ & - |A_1 \cap A_2| - \dots \qquad \text{all pairwise intersections} \\ & + |A_1 \cap A_2 \cap A_3| + \dots \qquad \text{all triple intersections} \\ & - \dots \end{aligned}$$

to find how many choices there are where at least one person gets his hat. This shoudl give you a series that you can recognize.