

MATH 10
ASSIGNMENT 15: SERIES 2
JAN 29, 2023

SERIES

Recall: given a sequence a_n , we define

$$\sum_{i=1}^{\infty} a_i = \lim S_n, \quad \text{where}$$
$$S_n = a_1 + \cdots + a_n = \sum_{i=1}^n a_i$$

(if this limit exists; otherwise we say that the series diverges and expression $\sum_{i=1}^{\infty} a_i$ is meaningless). For example:

$$1 + r + r^2 + \cdots = \sum_{i=0}^{\infty} r^i = \lim \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}, \quad |r| < 1$$

(this series is called the *geometric series*).

Note that it is quite possible that the sequence a_n converges but the sequence S_n of partial sums does not converge and thus the series $\sum_{i=1}^{\infty} a_i$ diverges!!

In the last HW, we have proved the following facts.

Theorem.

1. If a series $\sum a_n$ converges, then $\lim a_n = 0$. (Converse is not true: even if $\lim a_n = 0$, the series may diverge).
2. If $0 \leq a_n \leq b_n$, and $\sum b_n$ converges, then $\sum a_n$ also converges.

In fact, there is a more general result:

Theorem (Comparison test). If a_n, b_n are sequences such that $b_n \geq 0$, $|a_n| \leq b_n$ and the series $\sum_1^{\infty} b_n$ converges, then $\sum_1^{\infty} a_n$ also converges.

The proof of this result will be given later.

HOMEWORK

1. A tortoise is moving on the plane starting at the origin and then going 1 unit along the positive direction of x axis; then turning 90° to the left and going for 0.9 units, then turning 90° to the left and going for $(0.9)^2$ units, then....
 - (a) Show that if we consider the plane as the complex plane \mathbb{C} , then the position of the tortoise after n steps will be at the point $1 + r + r^2 + \cdots + r^{n-1}$, where $r = 0.9i$.
 - (b) Find where the tortoise will end up in the limit, after infinitely many steps.
2. Let a_n be a sequence such that $r = \lim \frac{|a_{n+1}|}{|a_n|}$ exists.
 - (a) Show that if $r > 1$, then $\lim a_n$ does not exist, and therefore $\sum_1^{\infty} a_n$ diverges (compare with Problem 1 from previous HW).
 - (b) Prove that if $r < 1$, then the series $\sum_1^{\infty} a_n$ converges. [Hint: compare with geometric series.]
 - (c) Give examples showing that if $r = 1$, then series $\sum_1^{\infty} a_n$ may converge or diverge. This is known as the *ratio test* for series convergence.
3. Prove that for any $x \in \mathbb{C}$, the series

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges. [Hint: use the previous problem.]

4. Let $E(x)$ be as defined in the previous problem. Prove that then $E(x+y) = E(x)E(y)$. [Hint: both sides can be written as “double series” $\sum a_{m,n}x^ny^m$. You can use without a proof that in all the series involved, rearranging the terms in any order will not affect the value of the series.]
5. Let $e = \sum_{n=0}^{\infty} \frac{1}{n!} = E(1)$. Prove that then $E(x) = e^x$:
- (a) For all integer x
 - (b) For all rational $x = p/q$
 - * (c) For all real x