MATH 6: HANDOUT 25
REVIEW TOURNAMENT I

1. Logic variables and truth tables

Logic variables are defined as variables that can only be either **True** or **False**. We can perform logical operations between two variables, and the result of these operations is another logical variable. Some examples of these operations are $A \text{ AND } B$, $A \text{ OR } B$, $A \Rightarrow B$, etc...

An easy way of understanding complicated operations between logic variables is by making truth tables. For example, the truth table of the operation OR looks like:

A	B	$A \text{ OR } B$
T	T	T
T	F	T
F	T	T
F	F	F

Problems.

1. Define a new logical operation, XOR (exclusive or) as follows: $A \text{ XOR } B$ is true if exactly one of A, B is true, and false otherwise.
   (a) (2 PTS.) Write the truth table for $A \text{ XOR } B$.
   (b) (3 PTS.) Express XOR using only AND, OR, and NOT (that is, write a formula equivalent to $A \text{ XOR } B$ using only AND, OR, and NOT).
   (c) (2 PTS.) Prove that your formula is correct by using a truth table.
2. (4 PTS.) Write truth tables for formulas $A \text{ AND}(B \text{ OR } C)$ and $(A \text{ AND } B) \text{ OR } C$ (hint: there will be 8 rows in the table). Argue if these formulas are equivalent or not.
3. (5 PTS.) Identify who is a knight and who is a knave in the following problem by using a truth table: On the island of Knights and Knaves, you meet three inhabitants: Bob, Mel and Peggy. Bob says that it's not true that Peggy is a knave. Mel says that Peggy is a knight or Bob is a knave. Peggy claims, "Both I am a knight and Bob is a knave."

2. Sets

Sets are collections of objects. Objects in the sets are regarded as elements of the sets. As with logical variables, we can do operations with sets. The most used operations are:

- $A \cup B$: union of A and B. It consists of all elements which are in either A or B (or both):
$$A \cup B = \{x \mid x \in A \text{ OR } x \in B\}.$$
- $A \cap B$: intersection of A and B. It consists of all elements which are in both A and B:
$$A \cap B = \{x \mid x \in A \text{ AND } x \in B\}.$$
- $\overline{A}$: complement of A, i.e. the set of all elements which are not in A:
$$\overline{A} = \{x \mid x \notin A\}.$$
- $|A|$: number of elements in a set A (if this set is finite).

Problems.

4. (a) (2 PTS.) Show that $|A \cup B| = |A| + |B| - |A \cap B|$.
   (b) (3 PTS.) Find an equivalent formula for $|A \cup B \cup C|$ which uses $|A|, |B|, |C|$, $|A \cap B|$.
   (c) (2 PTS.) Show that the formula you found is correct.
5. (1 PT. EACH) Let $A = [1, 3] = \{x \mid 1 \leq x \leq 3\}$, $B = \{x \mid x \geq 2\}$, $C = \{x \mid x \leq 1.5\}$. Describe $\overline{A}$, $\overline{B}$, $\overline{C}$, $A \cap B$, $A \cap C$, $A \cap (B \cup C)$, $A \cap B \cap C$.
6. (5 PTS.) Find sets A, B, C if you now that $A \cup B = \{1, 3, 4, 5, 7\}$, $B \cup C = \{1, 2, 4, 5, 6, 8, 9\}$, $(A \cup B) \cap C = \emptyset$, $(B \cup C) \cap A = \{1, 5\}$.

3. Permutations, combinations, and probability

Permutations count in how many ways we can choose k elements from a collection of n elements in a way in which the order matters and repetitions are not allowed. If we want to choose 4 cards from a deck of 52, we have 52 possibilities for the first card, 51 for the second, and so on. Therefore, we would have a total of

$$\underline{\quad 52 \quad} \times \underline{\quad 51 \quad} \times \underline{\quad 50 \quad} \times \underline{\quad 49 \quad} = \frac{51!}{48!} \quad \text{possibilities}$$

In general, permutations of k elements from a collection of n elements is given by ${}_nP_k = \frac{n!}{(n-k)!}$

If we have repeated elements, we can also find the **Permutations with repetition**. In this case, we need to make sure that we do not count more than once repeated objects. If we have an element that is repeated m times, then we would have that the number of permutations is given by

$$\frac{{}_nP_k}{m!}.$$

In other words, we first find the number of permutations assuming the elements are different, and then we correct for the repeated terms by counting in how many ways we can organize the same element m times.

Finally, a **combinations** counts in how many ways we can choose k elements from a collection of n elements if the order in which we choose it does not matter. In this case, we have the the combinations are given by

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{{}_nP_k}{k!}$$

The way to remember this is by thinking that the order matters first, and then you divide by counting in how many different ways you can rearrange your selection.

To calculate the **probability**, we use the basic probability rule:

$$P(win) = \frac{\text{number of winning outcomes}}{\text{total number of possible outcomes}}$$

Probabilities also satisfy the Complement Rule, which tells us that if the probability of an even happening is P, then the probability of that event **not** happening is $1 - P$.

If we do one trial right after the other and the second event does not depend on the outcome of the first one, then the probability of getting A in the first trial and B in the second one is given by

$$P(A, \text{then } B) = P(A)P(B).$$

Problems.

7. (a) (2 pts.) How many ways are there to draw 3 cards from a 52-card deck? (Order matters: drawing first king of spades, then queen of hearts is different from drawing them in opposite order).
(b) (2 pts.) How many ways are there to draw 3 cards from a 52-card deck if after each drawing we record the card we got, then return the card to the deck and reshuffle the deck? (As before, order matters.)
(c) (3 pts.) How many ways are there to draw 3 cards from the same deck if the order does not matter?

8. (4 pts.) In how many ways can you arrange the letters from the word TICKTOCK?

9. (a) (1 pt.) What is the probability that when we roll the die once, the number will be less than 5?
(b) (1 pt.) What is the probability that when we roll the die once, the number will be less than 7?
(c) (2 pts.) What is the probability that when we roll the die twice, at least one result will be a 6?
(d) (1 pts.) What is the probability that when we roll the die twice, at least one result will be a 7?
(e) (2 pts.) What is the probability that when we roll the die three times, all the results will be odd?

10. A family has 4 sons and 3 daughters.
(a) (3 pts.) In how many ways can they be seated on 7 chairs so that at least 2 boys are next to each other? (Hint: Use complement counting. In what case are there no boys sitting next to each other?)
(b) (3 pts.) If we assign their seats at random, what is the probability that two boys will be seating next to each other?

11. Imagine that we toss a coin 10 times.
(a) (2 pts.) What is the probability that all throws will land on heads?
(b) (2 pts.) What is the probability that all throws will land on tails?
(c) (3 pts.) What is the probability of getting at least one head?

4. Arithmetic and Geometric progressions

A sequence of numbers is an **arithmetic sequence** or **arithmetic progression** if the difference between consecutive terms is the same number, the **common difference** d. The first term of a sequence is usually denoted a_1. In general the n-th term is denoted as a_n. Some useful properties of arithmetic sequences are:

- If you know the starting point of the sequence a_1 and the common difference d, you can find any term in the sequence by
$$a_n = a_1 + (n-1)d$$
- Each term of the sequence is the arithmetic mean of its neighbors
$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$
- We can find the common difference if we know two terms of the sequence, as well as their relative position
$$d = \frac{a_m - a_n}{m - n}$$
- We can sum all the terms in an arithmetic sequence by doing
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = n \times \frac{a_1 + a_n}{2}$$

A sequence of numbers is a **geometric sequence** or **geometric progression** if if the next number in the sequence is the current number times a fixed constant called the **common ratio** q. Some useful properties of geometric sequences are

- If you know the starting point of the sequence a_1 and the common ratio q, you can find any term in the sequence by
$$a_n = a_1 q^{n-1}$$
- Each term of the sequence is the geometric mean of its neighbors
$$a_n = \sqrt{a_{n-1} \cdot a_{n+1}}$$
- We can sum all the terms in a geometric sequence by doing
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \frac{a_1(1-q^n)}{1-q}$$

Problems.

12. (3 PTS.) Find the common difference d and the first term a_1 in an arithmetic sequence if the 9-th term is 18 and the 11-th term is 44.

13. (3 PTS.) Find the sum of the first 100 odd numbers.

14. (3 PTS.) Suppose that you have terms a_m and a_n of a geometric sequence. Find the value of the common ratio q. You can assume that $m > n$.

15. (4 PTS.) Simplify the following expression
$$1 + x + x^2 + x^3 + \ldots + x^{100}$$

5. Factorization and systems of linear equations

To handle algebraic identities, it is very useful to be able to factorize them. Some useful identities are:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$
$$ab + ac = a(b+c)$$

Systems of linear equations are two or more linear equations that are using the same variables, hold true at the same time and have to solved together. One common method for solving systems of linear equations is the substitution method, in which we select the simpler equation and try to express one variable in terms of the other variable. Then, we substitute this variable in the other equation, and we solve it as a regular equation.

Problems.

16. (1 pt. each) Factor:

(a) $6a + 12 =$

(b) $mn + n =$

(c) $5xy - 15x =$

(d) $4ax - 8ax^2 + 12ax^3 =$

(e) $9 - x^2 =$

(f) $x^6 - 4 =$

(g) $9 - 6x + x^2 =$

(h) $a^3 - 2a^2x + ax^2 =$

17. (4 pts.) Solve the following system of linear equations:

$$\begin{cases} 5x + 2y = 16 \\ 2x + 3y = 13 \end{cases}$$

18. (4 pts.) The sum of two numbers is 27. Twice the larger number is 11 less than 3 times the smaller number. What are the two numbers?

19. (5 pts.) The sum of two numbers is $\frac{41}{35}$ and the difference is $\frac{1}{35}$. What are the two numbers?