

Homework for May 1, 2021.

### Algebra/Geometry. Complex numbers.

Please, complete the previous homework assignments. Review the classwork handout on complex numbers. Solve the following problems.

#### Problems.

- (i) find the magnitude and the argument, (ii) compute the inverse, and (iii) find the magnitude and the argument of the inverse for the following complex numbers:
  - $1 + i$
  - $-i$
  - $1 + ix$
  - $\frac{\sqrt{3}}{2} + \frac{i}{2}$
  - $\frac{1}{2-i} - \frac{1}{2+i}$
- Find a complex number  $z$  whose magnitude is 2 and the argument  $Arg(z) = \frac{\pi}{4} = 45^\circ$ .
- Draw the following sets of points on complex plane.
  - $\{z | Re(z) = 1\}$
  - $\{z | Arg(z) = \frac{3\pi}{4} = 135^\circ\}$
  - $\{z | |z| = 1\}$
  - $\{z | Re(z^2) = 0\}$
  - $\{z | |z^2| = 2\}$
  - $\{z | |z - 1| = 1\}$
  - $\{z | z + \bar{z} = 1\}$
- Prove that for any complex number  $z$ , we have
  - $|\bar{z}| = |z|, Arg(\bar{z}) = -Arg(z)$
  - $\frac{\bar{z}}{z}$  has magnitude 1; check this for  $z = 1 - i$ .
- If  $z$  has magnitude 2 and argument  $\frac{\pi}{2}$  and  $w$  has magnitude 3 and argument  $\frac{\pi}{3}$ , what will be the magnitude and the argument of  $zw$ ? Write it in the form  $a + bi$ .
- Let  $P(x)$  be a polynomial with real coefficients.
  - Prove that for any complex number  $z$ , we have  $\overline{P(z)} = P(\bar{z})$

- b. Let  $z$  be a complex root of this polynomial,  $P(z) = 0$ . Prove that then  $\bar{z}$  is also a root,  $P(\bar{z}) = 0$ .
7. Solve the equation  $x^3 - 4x^2 + 6x - 4 = 0$ . Find the sum and product of the roots in two ways: by using Vieta formulas and by explicit computation. Check that the results match.
8. Simplify following expression:
- $(1 + \sin \alpha)(1 - \sin \alpha)$
  - $(1 + \cos \alpha)(1 - \cos \alpha)$
  - $\sin^4 \alpha - \cos^4 \alpha$
9. Prove the following equalities:
- $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$
  - $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$
  - $\cos 4\alpha = 8 \cos^4 \alpha - 8 \cos^2 \alpha + 1$
  - $\sin 4\alpha = 4 \sin \alpha \cos^3 \alpha - 4 \cos \alpha \sin^3 \alpha$
  - $\sin 5\alpha = 16 \sin^5 \alpha - 20 \sin^3 \alpha + 5 \sin \alpha$
  - $\cos 5\alpha = \dots$  (find the expression)
10. Solve the following equation:
- $\cos^2 \pi x + 4 \sin \pi x + 4 = 0$
11. Solve the following equations and inequalities:
- $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$
  - $\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)$
  - $\sin^2 x - 2 \sin x \cos x = 3 \cos^2 x$
  - $\sin 6x + 2 = 2 \cos 4x$
  - $\cot x - \tan x = \sin x + \cos x$
  - $\sin x \geq \pi/2$
  - $\sin x \leq \cos x$

## Algebra: polynomials recap.

Review the classwork notes on polynomials, factorization and Vieta theorem. Solve the following problems; skip those you already solved.

### Problems.

- Write Vieta formulae for the cubic equation,  $x^3 + Px^2 + Qx + R = 0$ . Let  $x_1, x_2$  and  $x_3$  be the roots of this equation. Find the following combination in terms of  $P, Q$  and  $R$ ,
  - $(x_1 + x_2 + x_3)^2$
  - $x_1^2 + x_2^2 + x_3^2$
  - $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
  - $(x_1 + x_2 + x_3)^3$

- The three real numbers  $x, y, z$ , satisfy the equations

$$x + y + z = 6$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{6}$$

$$xy + yz + zx = 11$$

- Find a cubic polynomial whose roots are  $x, y, z$
  - Find  $x, y, z$
- Find two numbers  $u, v$  such that

$$u + v = 6$$

$$uv = 13$$

- Find three numbers,  $a, b, c$ , such that

$$a + b + c = 2$$

$$ab + bc + ca = -7$$

$$abc = -14$$

- Find all real roots of the following polynomial and factor it.

- a.  $x^8 + x^4 + 1$
  - b.  $x^4 - x^3 + 5x^2 - x - 6$
  - c.  $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$
6. Perform the long division, finding the quotient and the remainder, on the following polynomials.
- a.  $(x^3 - 3x^2 + 4) \div (x^2 + 1)$
  - b.  $(x^3 - 3x^2 + 4) \div (x^2 - 1)$