MATH 8: ASSIGNMENT 14: EUCLIDEAN GEOMETRY 2

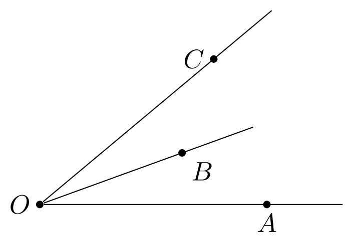
JANUARY 17, 2021

1. Reminder: Postulates and previously proved results

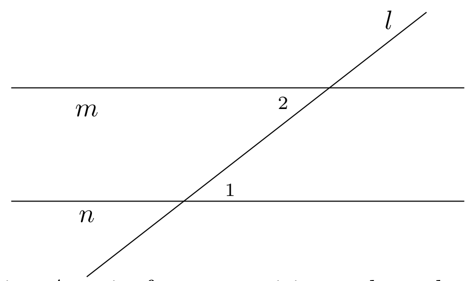
Axiom 1. *For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

Axiom 2. *If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*



Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*



In addition, we will assume that given a line l and a point A on it, for any positive real number d, there are exactly two points on l at distance d from A, on opposite sides of A, and similarly for angels: given a ray and angle measure, there are exactly two angles with that measure having that ray as one of the sides.

Theorem 1. *If distinct lines l, m intersect, then they intersect at exactly one point.*

Theorem 2. *Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.*

Theorem 3. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Theorem 4. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in the figure below (such a pair of angles are called vertical). Then $m\angle 1 = m\angle 3$.*

Theorem 5. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are perpendicular and write $l \perp m$.)*

Theorem 6. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

Theorem 8. *Given any three points A, B, C, which are not on the same line, and line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, we have $m\angle ABC + m\angle BCA + m\angle CAB = 180°$. (Such a figure of three points and their respective line segments is called a triangle, written $\triangle ABC$. The three respective angles are called the triangle's interior angles.)*

2. Congruence

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D, then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong \overline{CD}$.

- If two triangles $\triangle ABC$, $\triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Note that congruence of triangles is sensitive to which vertices on one triangle correspond to which vertices on the other. Thus, $\triangle ABC \cong \triangle DEF \implies \overline{AB} \cong \overline{DE}$, and it can happen that $\triangle ABC \cong \triangle DEF$ but $\neg(\triangle ABC \cong \triangle EFD)$.

3. Congruence of Triangles

Triangles consist of six pieces (three line segments and three angles), but some notion of constancy of shape in triangles is important in our geometry. We describe below some rules that allow us to, in essence, uniquely determine the shape of a triangle by looking at a specific subset of its pieces.

Axiom 5 (SAS Congruence)**.** *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Other congruence rules about triangles follow from the above: the ASA and SSS rules. However, their proofs are less interesting than other problems about triangles, so we can take them as axioms and continue.

Axiom 6 (ASA Congruence)**.** *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

Axiom 7 (SSS Congruence)**.** *If two triangles have three sides congruent, then the triangles are congruent.*

4. Isosceles triangles

A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

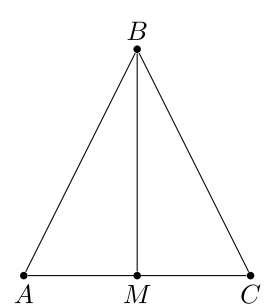
Theorem 9 (Base angles equal)**.** *If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.*
Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

A proof is given to you as homework.

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from A is perpendicular to BC (it exists and is unique by Theorem 7); the median from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC); and the angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide.

Theorem 10. *If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.*



Proof. Consider triangles $\triangle ABM$ and $\triangle CBM$. Then $AB = CB$ (by definition of isosceles triangle), $AM = CM$ (by definition of midpoint), and side BM is the same in both triangles. Thus, by SAS axiom, $\triangle ABM \cong \triangle CBM$. Therefore, $m\angle ABM = m\angle CBM$, so BM is the angle bisector.
Also, $m\angle AMB = m\angle CMB$. On the other hand, $m\angle AMB + m\angle CMB = m\angle AMC = 180^\circ$. Thus, $m\angle AMB = m\angle CMB = 180^\circ/2 = 90^\circ$. □

5. Triangle inequalities

In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal (Theorem 9). The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

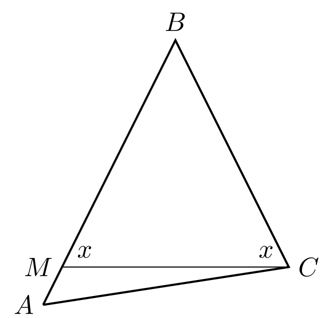
Theorem 11. *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Proof. Assume not. Then either $BC = AB$ or $BC < AB$.

But if $BC = AB$, then $\triangle ABC$ is isosceles, so by Theorem 9, $m\angle A = m\angle C$ as base angles, which gives a contradiction.

Now assume $BC < AB$, find the point M on AB so that $BM = BC$, and draw the line MC. Then $\triangle MBC$ is isosceles, with apex at B. Hence $m\angle BMC = m\angle MCB$ (these two angles are denoted by x in the figure.) On one hand, $m\angle C > x$ (this easily follows from Axiom 3). On the other hand, since x is an external angle of $\triangle AMC$, by Problem 1 we have $x > m\angle A$. These two inequalities imply $m\angle C > m\angle A$, which contradicts what we started with.

Thus, assumptions $BC = AB$ or $BC < AB$ both lead to a contradiction.



□

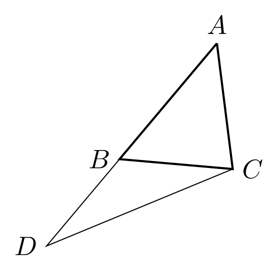
The converse of the previous theorem is also true: opposite a longer side, there must be a larger angle.The proof is left as an exercise.

Theorem 12. *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

Theorem 13 (The triangle inequality)**.** *In $\triangle ABC$, we have $AB + BC > AC$.*

Proof. Extend the line AB past B to the point D so that $BD = BC$, and join the points C and D with a line so as to form the triangle ADC. Observe that $\triangle BCD$ is isosceles, with apex at B; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 11, this implies $AD > AC$. Our result now follows from $AD = AB + BD$ (Axiom 2) □



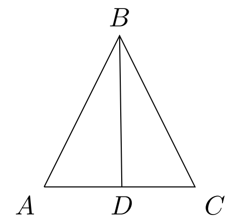
6. Homework

Note that you may use all results that are presented in the previous sections. This means that you may use Theorem 3, for example, if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

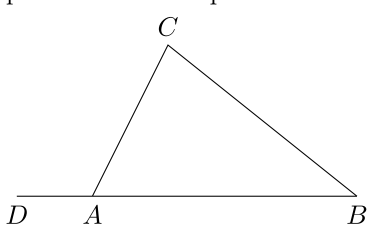
1. (This problem is from last week) A triangle in which two sides are congruent is called *isosceles.* Such triangles have many special properties.

(a) Let $\triangle ABC$ be an isosceles triangle, with $\overline{AB} \cong \overline{BC}$. Suppose D is a point on $\overline{AC}$ such that $\overline{AD} \cong \overline{DC}$ (such point is called *midpoint* of the segment). Prove that then, $\triangle BD \cong \triangle CBD$ and deduce from this that $\angle DBA \cong \angle DBC$, and $\angle A \cong \angle C$. What can we say about $\angle ADB$?

(b) Conversely, show that if $\triangle ABC$ is such that $\angle A \cong \angle C$, then $\triangle ABC$ is isosceles, with $\overline{AB} \cong \overline{BC}$.



2. Given a triangle $\triangle ABC$, let D be a point on the line AB, so that A is between D and B. In this situation, angle $\angle DAC$ is called an *external angle* of $\triangle ABC$. Prove that $m\angle DAC = m\angle B + m\angle C$ (in particular this implies that $m\angle DAC > m\angle B$, and similarly for $\angle C$).



3. (Perpendicular bisector) Let $\overline{AB}$ be a line segment. The perpendicular bisector L of $\overline{AB}$ is the line that passes through the midpoint M of $\overline{AB}$ and is perpendicular to $\overline{AB}$.

(a) Prove that for any point P on L, triangles $\triangle APM$ and $\triangle BPM$ are congruent. Deduce from this that $AP = BP$.

(b) Conversely, let P be a point on the plane such that $AP = BP$. Prove that then P must be on L.

Taken together, these two statements say that a point P is *equidistant* from A, B if and only if it lies on the perpendicular bisector L of segment $\overline{AB}$. Another way to say it is that the *locus* of all the points equidistant from A, B is the perpendicular bisector of $\overline{AB}$.

4. Show that for any triangle $\triangle ABC$, the perpendicular bisectors of the three sides intersect at a single point, and this point is equidistant from all three vertices of the triangle. [Hint: consider the point where two of the bisectors intersect. Prove that this point is equidistant from all three vertices.]

Note: the intersection point can be outside the triangle.

5. (Slant lines and perpendiculars) Let P be a point not on line l, and let $Q \in l$ be such that $PQ \perp l$.

Prove that then, for any other point R on line l, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line.

Note: you can not use the Pythagorean theorem for this, as we haven't yet proved it!

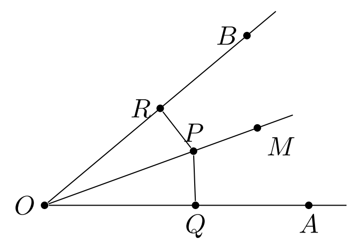
Instead, use Theorem 11.

6. (Angle bisector). Define a distance from a point P to line l as the length of the perpendicular from P to l (compare with the previous problem).

Let $\overrightarrow{OM}$ be the angle bisector of $\angle AOB$, i.e. $\angle AOM \cong \angle MOB$.

(a) Let P be any point on $\overrightarrow{OM}$, and PQ, PR – perpendiculars from P to sides $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively. Use ASA axiom to prove that triangles $\triangle OPR$, $\triangle OPQ$ are congruent, and deduce from this that distances from P to $\overrightarrow{OA}$, $\overrightarrow{OB}$ are equal.

(b) Prove that conversely, if P is a point inside angle $\angle AOB$, and distances from P to the two sides of the angle are equal, then P must lie on the angle bisector $\overrightarrow{OM}$



These two statements show that the locus of points equidistant from the two sides of an angle is the angle bisector

7. Prove that in any triangle, the three angle bisectors intersect at a single point (compare with problem 4)