Homework for April 28, 2019.

Algebra/Geometry. Complex numbers.

Please, complete the previous homework assignments. Review the classwork handout on complex numbers. Solve the following problems.

Problems.

1. (i) find the magnitude and the argument, (ii) compute the inverse, and (iii) find the magnitude and the argument of the inverse for the following complex numbers:
   a. $1 + i$
   b. $-i$
   c. $1 + ix$
   d. $\frac{\sqrt{3}}{2} + \frac{i}{2}$
   e. $\frac{1}{2-i} - \frac{1}{2+i}$
2. Find a complex number z whose magnitude is 2 and the argument $Arg(z) = \frac{\pi}{4} = 45°$.
3. Draw the following sets of points on complex plane.
   a. $\{z|Re(z) = 1\}$
   b. $\left\{z|Arg(z) = \frac{3\pi}{4} = 135°\right\}$
   c. $\{z|\ |z| = 1\}$
   d. $\{z|Re(z^2) = 0\}$
   e. $\{z|\ |z^2| = 2\}$
   f. $\{z|\ |z - 1| = 1\}$
   g. $\{z|\ z + \bar{z} = 1\}$
4. Prove that for any complex number z, we have
   a. $|\bar{z}|=|z|$, $Arg(\bar{z}) = -Arg(z)$
   b. $\frac{\bar{z}}{z}$ has magnitude 1; check this for $z = 1 - i$.
5. If z has magnitude 2 and argument $\frac{\pi}{2}$ and w has magnitude 3 and argument $\frac{\pi}{3}$, what will be the magnitude and the argument of zw? Write it in the form $a + bi$.
6. Let $P(x)$ be a polynomial wit real coefficients.
   a. Prove that for any complex number z, we have $\overline{P(z)} = P(\bar{z})$

b. Let z be a complex root of this polynomial, $P(z) = 0$. Prove that then $\bar{z}$ is also a root, $P(\bar{z}) = 0$.

7. Solve the equation $x^3 - 4x^2 + 6x - 4 = 0$. Find the sum and product of the roots in two ways: by using Vieta formulas and by explicit computation. Check that the results match.
8. Simplify following expression:
   a. $(1 + \sin\alpha)(1 - \sin\alpha)$
   b. $(1 + \cos\alpha)(1 - \cos\alpha)$
   c. $\sin^4\alpha - \cos^4\alpha$
9. Prove the following equalities:
   a. $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$
   b. $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$
   c. $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$
   d. $\sin 4\alpha = 4\sin\alpha\cos^3\alpha - 4\cos\alpha\sin^3\alpha$
   e. $\sin 5\alpha = 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha$
   f. $\cos 5\alpha = \cdots$ (find the expression)
10. Solve the following equation:
    a. $\cos^2\pi x + 4\sin\pi x + 4 = 0$
11. Solve the following equations and inequalities:
    a. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$
    b. $\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)$
    c. $\sin^2 x - 2\sin x\cos x = 3\cos^2 x$
    d. $\sin 6x + 2 = 2\cos 4x$
    e. $\cot x - \tan x = \sin x + \cos x$
    f. $\sin x \geq \pi/2$
    g. $\sin x \leq \cos x$

Algebra: polynomials recap.

Review the classwork notes on polynomials, factorization and Vieta theorem. Solve the following problems; skip those you already solved.

Problems.

1. Write Vieta formulae for the cubic equation, $x^3 + Px^2 + Qx + R = 0$. Let x_1, x_2 and x_3 be the roots of this equation. Find the following combination in terms of P, Q and R,
   a. $(x_1 + x_2 + x_3)^2$
   b. ${x_1}^2 + {x_2}^2 + {x_3}^2$
   c. $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$
   d. $(x_1 + x_2 + x_3)^3$
2. The three real numbers x, y, z, satisfy the equations

$$x + y + z = 6$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{11}{6}$$

$$xy + yz + zx = 11$$

   a. Find a cubic polynomial whose roots are x, y, z
   b. Find x, y, z
3. Find two numbers u, v such that

$$u + v = 6$$

$$uv = 13$$

4. Find three numbers, a, b, c, such that

$$a + b + c = 2$$

$$ab + bc + ca = -7$$

$$abc = -14$$

5. Find all real roots of the following polynomial and factor it.

a. $x^8 + x^4 + 1$
b. $x^4 - x^3 + 5x^2 - x - 6$
c. $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$

6. Perform the long division, finding the quotient and the remainder, on the following polynomials.
   a. $(x^3 - 3x^2 + 4) \div (x^2 + 1)$
   b. $(x^3 - 3x^2 + 4) \div (x^2 - 1)$