April 14, 2019 Math 9

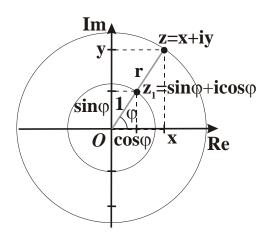
Algebra.

Trigonometric form of complex numbers. Geometric interpretation.

Let us consider complex numbers with the absolute value of 1,

$$z_1 = x_1 + iy_1$$
, $|z_1|^2 = z_1 \overline{z_1} = x_1^2 + y_1^2 = 1$.

There is an obvious one-to-one correspondence between such numbers and points $Z_1(x_1,y_1)$ on a circle of unit radius. Hence, we can express such numbers in terms of an angle, φ , parameterizing points on the unit circle,



$$z_1 = x_1 + iy_1 = \cos \varphi + i \sin \varphi.$$

More generally, any complex number, z = x + iy, whose absolute value is $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2} = r$, can be written in the trigonometric form as, $z = x + iy = r(\cos \varphi + i \sin \varphi)$.

Geometrically, it is represented by a point Z(x,y) on a circle of radius r=|z|. Position of this point is specified by an angle, φ , which is conventionally measured counterclockwise from the positive direction of the X-axis. Angle φ is called the argument of the complex number z and is denoted $\varphi = Arg(z)$. Thus, instead of describing a complex number by its real and imaginary part, i.e. its coordinates, (x,y), we can describe it by its magnitude and argument (polar coordinates), (r,φ) , where $r \geq 0$ and $0 \leq \varphi = Arg(z) < 360^\circ$.

It is now easy to prove the following important property of the multiplication of complex numbers.

Theorem. When we multiply two complex numbers, magnitudes multiply and arguments add,

$$|z_1 z_2| = |z_1||z_2|$$
, $Arg(z_1 z_2) = (Arg(z_1) + Arg(z_2)) \mod 360^\circ$.

Proof. Let $Arg(z_1) = \varphi_1$ and $Arg(z_2) = \varphi_2$, so $z_1 = |z_1|(\cos \varphi_1 + i \sin \varphi_1)$ and $z_2 = |z_2|(\cos \varphi_2 + i \sin \varphi_2)$. Perform the multiplication directly,

$$z_1 z_2 = |z_1|(\cos \varphi_1 + i \sin \varphi_1)|z_2|(\cos \varphi_2 + i \sin \varphi_2) =$$

$$|z_1||z_2|(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2))$$

= $|z_1||z_2|(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$

Complex numbers whose arguments would differ by multiples of 360° are identical and correspond to the same point on the complex plane. Hence, the argument is computed $mod~360^\circ$, ensuring that $0 \le \varphi_1 + \varphi_2 < 360^\circ$.

Theorem. Multiplication of a complex number, $z = x + iy = r(\cos \varphi + i \sin \varphi)$, by a complex number of unit magnitude and argument ψ ,

$$z_{\psi} = x_{\psi} + iy_{\psi} = \cos \psi + i \sin \psi,$$

corresponds to a counterclockwise rotation of the point, Z(x, y), on the complex plane, by an angle ψ ,

$$|zz_{\psi}| = r$$
, $Arg(zz_{\psi}) = \varphi + \psi$.

Proof. Indeed, perform the multiplication directly,

$$zz_{\psi} = (x + iy)(x_{\psi} + iy_{\psi}) = r(\cos\varphi + i\sin\varphi)(\cos\psi + i\sin\psi)$$
$$= r(\cos\varphi\cos\psi - \sin\varphi\sin\psi + i(\sin\varphi\cos\psi + \cos\varphi\sin\psi))$$
$$= r(\cos(\varphi + \psi) + i\sin(\varphi + \psi))$$

It is clear that multiplication by a complex number with magnitude r' and argument ψ is equivalent to the combination of multiplication by a number of unit magnitude and argument ψ , and y the real number r'.

Theorem. Multiplication of a complex number, z = x + iy, by a complex number with magnitude r' and argument ψ ,

$$w = r'(\cos \psi + i \sin \psi),$$

results in a point on the complex plane, which is obtained from the point Z(x,y) by the combination of a rotation by angle ψ and a homothety (rescaling) with scale factor r'.

Multiplication of all complex numbers by a complex number $w = r'(\cos \psi + i \sin \psi)$ is a transformation of the complex plane, which maps complex plane on itself. Identifying multiplication by a complex number with such transformation, we can state the following.

Theorem. Multiplication by a complex number with magnitude r' and argument ψ , $w = r'(\cos \psi + i \sin \psi)$, is a combination of rotation by angle ψ and homothety (rescaling) with scale factor r'.

De Moivre's formula.

Theorem. The formula named after Abraham de Moivre states that for any complex number, $z = x + iy = r(\cos \varphi + i \sin \varphi)$, and for any integer $n \in \mathbb{N}$,

$$z^{n} = (r(\cos \varphi + i \sin \varphi))^{n} = r^{n}(\cos n\varphi + i \sin n\varphi)$$

Proof 1 (Mathematical induction).

- 1. Base case, n = 1: $z^1 = r(\cos \varphi + i \sin \varphi)$ is true.
- 2. I(n) = I(n+1). Assume $z^n = r^n(\cos n\varphi + i\sin n\varphi)$ is true. Then,

$$z^{n+1} = z \cdot z^n = r(\cos \varphi + i \sin \varphi) \cdot (r(\cos \varphi + i \sin \varphi))^n$$

$$= r(\cos \varphi + i \sin \varphi)r^n(\cos n\varphi + i \sin n\varphi)$$

$$= r^{n+1}(\cos \varphi \cos n\varphi - \sin \varphi \sin n\varphi)$$

$$+ i(\sin \varphi \cos n\varphi + \cos \varphi \sin n\varphi))$$

$$= r^{n+1}(\cos(n+1)\varphi + i \sin(n+1)\varphi)$$

Proof 2 (Geometrical).

$$z^{n} = (r(\cos\varphi + i\sin\varphi))^{n} = r(\cos\varphi + i\sin\varphi)r(\cos\varphi + i\sin\varphi)\dots r(\cos\varphi + i\sin\varphi)$$

By property of the multiplication of complex numbers, absolute values multiply, while arguments add. Therefore,

 $|z^n| = |z|^n = r^n$, and $Arg(z^n) = nArg(z) = n\varphi$, wherefrom it follows that $z^n = r^n(\cos n\varphi + i\sin n\varphi)$.

n-th root.

The formula of de Moivre allows us to compute n -th root of a complex number. Suppose we want to solve the equation,

$$w^n = z$$

where $w, z \in \mathbb{C}$, so w is the n-th root of z. According to de Moivre's formula, if $w = |w|(\cos \psi + i \sin \psi)$, then $w^n = |w|^n(\cos n\psi + i \sin n\psi)$. Denoting $z = r(\cos \varphi + i \sin \varphi)$, we can rewrite the equation as,

$$w^n = |w|^n(\cos n\psi + i\sin n\psi) = r(\cos \varphi + i\sin \varphi)$$

One obvious solution is $r=|w|^n$ and $\varphi=n\psi, w=\sqrt[n]{r}\left(\cos\frac{\varphi}{n}+i\sin\frac{\varphi}{n}\right)$. However, because $\varphi=Arg(z)=Arg(w^n)$ and $\psi=Arg(w)$ are determined modulo 360° (2π radians), there are other solutions, too, satisfying the above equation, such as $w=\sqrt[n]{r}\left(\cos\frac{\varphi+2\pi}{n}+i\sin\frac{\varphi+2\pi}{n}\right)$. Generally, we must have $r=|w|^n$, and $\varphi=Arg(z)=Arg(w^n)\ mod\ 360^\circ=nArg(w)\ mod\ 360^\circ=n\psi$ mod 360° . Altogether, there are n solutions,

$$w = \sqrt[n]{r} \left(\cos \frac{\varphi + 2\pi k}{n} + i \sin \frac{\varphi + 2\pi k}{n} \right), 0 \le k < n$$

This is a special case of the following extremely important result, called the fundamental theorem of algebra.

Theorem. Any polynomial with complex coefficients of degree n has exactly n roots (counting with multiplicities).

There is no simple proof of this theorem (and, in fact, no purely algebraic proof: all the known proofs use some geometric arguments).

In particular, since any polynomial with real coefficients can be considered as a special case of a polynomial with complex coefficients, this shows that any real polynomial of degree n has exactly n complex roots.