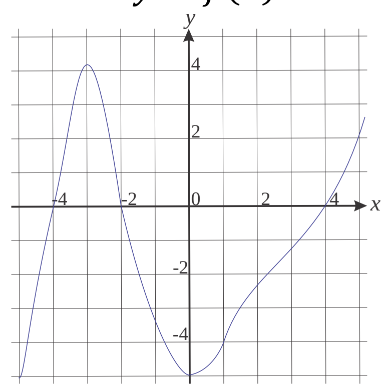
Homework for March 31, 2019.

Algebra.

Review the classwork handout. Complete the unsolved problems from the previous homework and classwork exercises. Solve the following problems.

1. From the picture, find in which interval(s) the function $y = f(x)$
   a. is monotonic
   b. has the same sign



2. Find all possible values of a such that equation $x^2 + ax + 9 = 0$ has two different roots, both of which are less than -1.
3. Draw graphs of the following functions
   a. $y = \left|\frac{1}{x-2} + 1\right|$
   b. $y = \frac{1}{|x|-2} + 1$
4. Find all x for which,
   a. $\sin x \cos x = \frac{1}{2}$
   b. $\sin x \cos x = \frac{\sqrt{3}}{2}$
5. Find the sum of the following series,

$$S = \cos x + \cos 3x + \cos 5x + \cos 7x + \cdots + \cos 2017x$$

(hint: multiply the sum by $2\sin x$)

6. Calculate:
   a. $\cos 75° + \cos 15° =$
   b. $\cos\frac{\pi}{12} - \cos\frac{5\pi}{12} =$
7. Prove the following equalities:
   a. $\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha} = \cot\frac{\alpha}{2}$
   b. $\sin^2\left(\frac{7\pi}{8} - 2\alpha\right) - \sin^2\left(\frac{9\pi}{8} - 2\alpha\right) = \frac{\sin 4\alpha}{\sqrt{2}}$
   c. $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}$

d. $\frac{\cot^2 2\alpha - 1}{2\cot 2\alpha} - \cos 8\alpha \cot 4\alpha = \sin 8\alpha$

e. $\sin^6 \alpha + \cos^6 \alpha + 3\sin^2 \alpha \cos^2 \alpha = 1$

f. $\frac{\sin 6\alpha + \sin 7\alpha + \sin 8\alpha + \sin 9\alpha}{\cos 6\alpha + \cos 7\alpha + \cos 8\alpha + \cos 9\alpha} = \tan\frac{15\alpha}{2}$

g. $\sin^6 \alpha + \cos^6 \alpha = \frac{5 + 3\cos 4\alpha}{8}$

h. $16\sin^5 \alpha - 20\sin^3 \alpha + 5\sin\alpha = \sin 5\alpha$

i. $\frac{\cos 64° \cos 4° - \cos 86° \cos 26°}{\cos 71° \cos 41° - \cos 49° \cos 19°}$

j. $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

k. $\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°} = 4$

8. Simplify the following expressions:

l. $\sin^2\left(\frac{\alpha}{2} + 2\beta\right) - \sin^2\left(\frac{\alpha}{2} - 2\beta\right)$

m. $2\cos^2 3\alpha + \sqrt{3}\sin 6\alpha - 1$

n. $\cos^4 2\alpha - 6\cos^2 2\alpha \sin^2 2\alpha + \sin^4 2\alpha$

o. $\sin^2(135° - 2\alpha) - \sin^2(210° - 2\alpha) - \sin^2 195° \cos(165° - 4\alpha)$

p. $\frac{\cos 2\alpha - \cos 6\alpha + \cos 10\alpha - \cos 14\alpha}{\sin 2\alpha + \sin 6\alpha + \sin 10\alpha + \sin 14\alpha}$

9. Let A, B and C be angles of a triangle. Prove that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

10. Solve the following equations and inequalities:

q. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$
r. $\cos 3x - \sin x = \sqrt{3}(\cos x - \sin 3x)$
s. $\sin^2 x - 2\sin x \cos x = 3\cos^2 x$
t. $\sin 6x + 2 = 2\cos 4x$
u. $\cot x - \tan x = \sin x + \cos x$
v. $\sin x \geq \pi/2$
w. $\sin x \leq \cos x$

Geometry.

Review the classwork notes. Solve the remaining problems from the previous homework and classwork exercises. Solve the following problems (problems with star are optional).

Problems.

1. Consider all possible configurations of the Apollonius problem (i.e. different possible choices of circles, points and lines). How many possibilities are there? Make the corresponding drawings and write the equations for finding the Apollonius circle in one of them (of your choice).
2. Find the equation of the locus of points equidistant from two lines, $y = ax + b$ and $y = mx + n$, where a, b, m, n are real numbers.
3. Using the method of coordinates, prove that the geometric locus of points from which the distances to two given points have a given ratio, $q \neq 1$, is a circle.
4. Prove that the locus of points $P(x, y)$ described by the equation $x^2 + 4y^2 = 6x + 8y$ is an ellipse. Find its center, major and minor half-axes, focal distance and foci.
5. (Skanavi 15.101) Given two points, M and N, with the co-ordinates (x_1, y_1) and (x_2, y_2), respectively, write the equation of the line with respect to which these two points are symmetric.
6. (Skanavi 15.103) Write an equation of the circle inscribed in the triangle formed by the lines $x = 0$, $y = 0$, and $3x + 4y - 12 = 0$.
7. (Skanavi 15.114) Write an equation of the circle passing through the origin, (0,0), the point (1,0), and tangent to the circle, $x^2 + y^2 - 9 = 0$.
8. (Skanavi 15.121) Write an equation of the circle tangent to the lines $y = 0$, $y = 4$, and $x + y + 1 = 0$.
9. (Skanavi 15.123) Write equations of the lines passing through the point (5,0) and tangent to the circle, $x^2 + y^2 - 9 = 0$.
10. *Prove that a square has the maximum area of all rectangles having the same perimeter.
11. *Prove that the circumradius of a triangle, R, is at least twice the inradius, r.