Geometry. Trigonometry.

Trigonometric formulas and equations.

Using the formulas for the sine and cosine of the sum/difference of two angles, which we have previously derived,

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

It is easy to obtain all other trigonometric formulae.

Exercise. Derive the following expressions for the products of sine and cosine,

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$$

Solution. These expressions are obtained by adding and subtracting the above expressions for $\sin(\alpha \pm \beta)$, $\cos(\alpha \pm \beta)$. For example,

$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$, etc.

Exercise. Derive the following expressions for sums and differences of sine and cosine,

$$\sin\alpha + \sin\beta = 2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$$

$$\sin\alpha - \sin\beta = 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\sin\left[\frac{1}{2}(\alpha-\beta)\right]$$

$$\cos\alpha + \cos\beta = 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right]$$

$$\cos\alpha - \cos\beta = -2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\sin\left[\frac{1}{2}(\alpha-\beta)\right]$$

Solution. The above expressions are obtained by representing α and β as, $\alpha = \frac{1}{2}(\alpha+\beta)+\frac{1}{2}(\alpha-\beta), \alpha = \frac{1}{2}(\alpha+\beta)-\frac{1}{2}(\alpha-\beta)$, and using the previously obtained expressions for $\sin(\alpha \pm \beta),\ \cos(\alpha \pm \beta)$.

Exercise. Derive the following expressions,

$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha)\cos(\beta)} \qquad \cot\alpha \pm \cot\beta = \pm\frac{\sin(\alpha \pm \beta)}{\sin(\alpha)\sin(\beta)}$$

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha) \qquad \cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \qquad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha) \qquad \cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1-\cos\alpha)} \qquad \cos\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1+\cos\alpha)}$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$\cot\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{1-\sin\alpha}} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}$$

$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$$

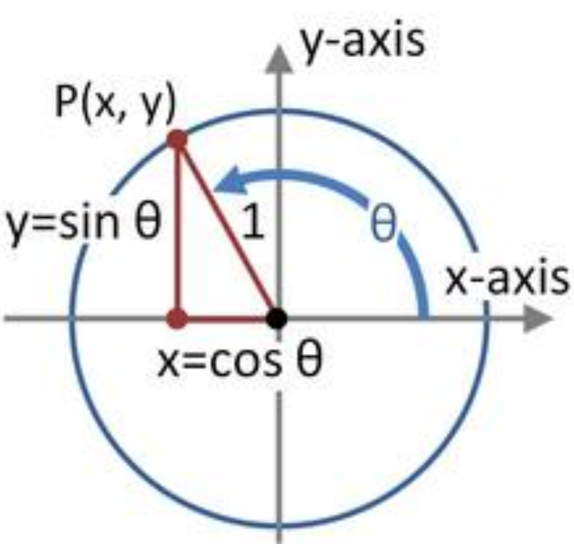
$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

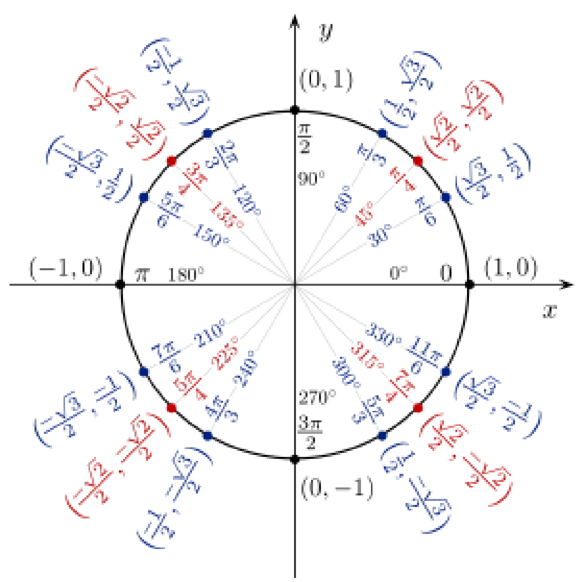
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

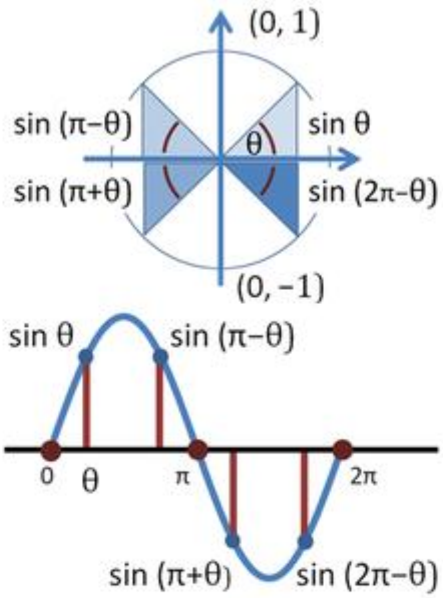
Trigonometric functions and relations.

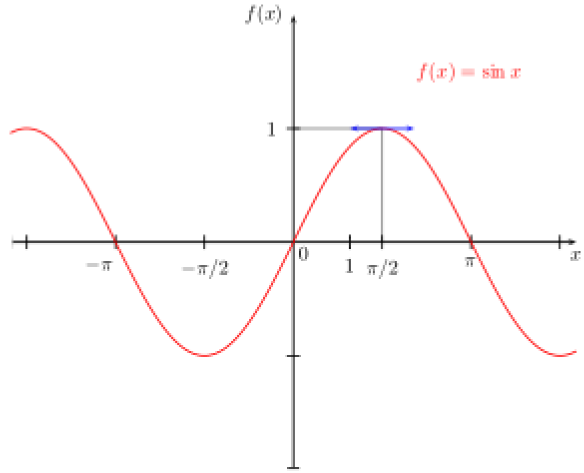
Exercise. Fill in the table of the trigonometric functions of complementary angles below.

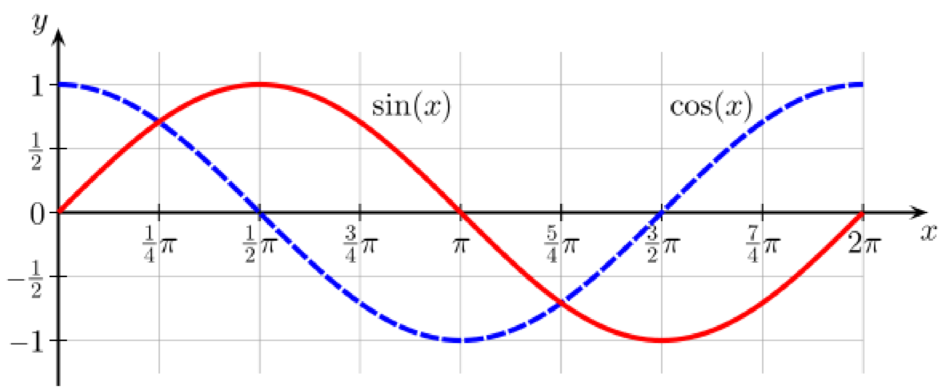
α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$
$\frac{\pi}{2}-\alpha$	$\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$			
$\frac{\pi}{2}+\alpha$				
$\pi-\alpha$				
$\pi+\alpha$				
$\frac{3}{2}\pi-\alpha$				
$\frac{3}{2}\pi+\alpha$				
$-\alpha$				






(0, 1)
sin (π−θ)
sin θ
θ
sin (π+θ)
sin (2π−θ)
(0, −1)
sin θ
sin (π−θ)
0
θ
π
2π
sin (π+θ)
sin (2π−θ)

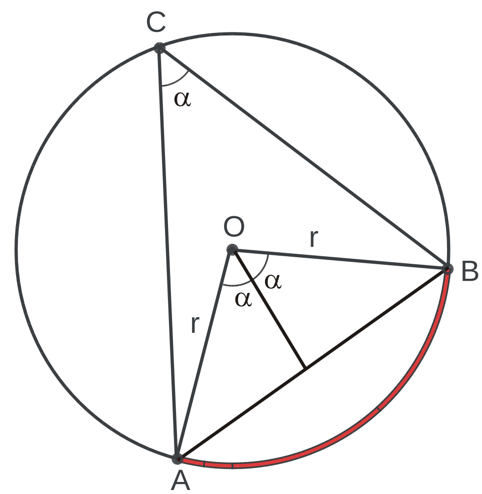

$f(x)$
$f(x) = \sin x$
1
$-\pi$
$-\pi/2$
0
1
$\pi/2$
π
x


y
1
$\frac{1}{2}$
0
$-\frac{1}{2}$
-1
$\sin(x)$
$\cos(x)$
$\frac{1}{4}\pi$
$\frac{1}{2}\pi$
$\frac{3}{4}\pi$
π
$\frac{5}{4}\pi$
$\frac{3}{2}\pi$
$\frac{7}{4}\pi$
2π
x

Homework review.

Problems.

3. Show that the length of a chord in a circle of unit diameter is equal to the sine of its inscribed angle.



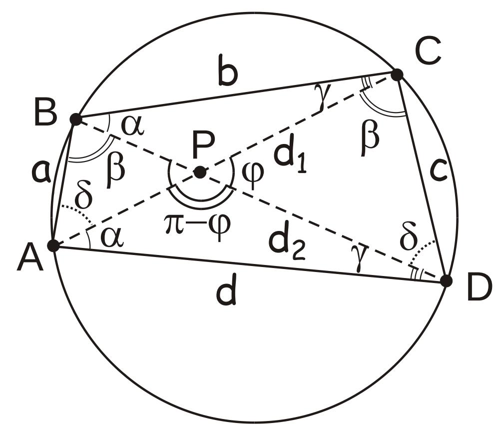
4. Using the result of the previous problem, express the statement of the Ptolemy theorem in the trigonometric form, also known as Ptolemy identity (see Figure):

   $\sin(\alpha+\beta)\sin(\beta+\gamma) = \sin\alpha\sin\gamma + \sin\beta\sin\delta,$

   if $\alpha+\beta+\gamma+\delta=\pi$.

5. Prove the Ptolemy identity in Problem 2 using the addition formulas for sine and cosine.

Solutions.



3. Consider the figure on the right,

$|AB| = 2r\sin\alpha = \sin\alpha$, if $d = 2r = 1$.

4. According to Ptolemy's theorem for a quadrilateral inscribed in a circle,

$d_1 d_2 = ac + bd$.

Applying this for the circle of the unit diameter and using the result of the previous problem, we obtain,

$\sin(\alpha+\beta)\sin(\beta+\gamma) = \sin\alpha\sin\gamma + \sin\beta\sin\delta$, where $\alpha+\beta$ and $\gamma+\delta$ are the opposite angles of an inscribed quadrilateral (and so are $\alpha+\delta$ and $\beta+\gamma$), and therefore $\alpha+\beta+\gamma+\delta=\pi$.

5. Using the multiplication formulas for sines we obtain,

$\sin\alpha\sin\gamma + \sin\beta\sin\delta = \frac{1}{2}[\cos(\alpha-\gamma) - \cos(\alpha+\gamma) + \cos(\beta-\delta) - \cos(\beta+\delta)] = \frac{1}{2}\left[\cos(\alpha-\gamma) + \cos(\beta-\delta) - \left(\cos(\alpha+\gamma) + \cos(\beta+\delta)\right)\right]$

$= \frac{1}{2}\left[2\cos\left(\frac{\alpha-\gamma+\beta-\delta}{2}\right)\cos\left(\frac{\alpha-\gamma-\beta+\delta}{2}\right)\right] = \cos\left(\frac{2\alpha+2\beta-\pi}{2}\right)\cos\left(\frac{\pi-2\gamma-2\beta}{2}\right) =$
$\sin(\alpha+\beta)\sin(\beta+\gamma)$.

6. Using the Sine and the Cosine theorems, prove the Hero's formula for the area of a triangle,

$$S_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$ is the semi-perimeter.

Solution. The area of a triangle ABC is $S_{\Delta ABC} = \frac{1}{2}ab\sin\gamma$, so

$$\sin^2\gamma = \frac{4{S_{\Delta ABC}}^2}{a^2b^2}$$

From the Law of cosines, we have

$$\cos^2\gamma = \left(\frac{a^2+b^2-c^2}{2ab}\right)^2$$

Adding the two expressions, we obtain, $1 = \frac{4{S_{\Delta ABC}}^2}{a^2b^2} + \frac{(a^2+b^2-c^2)^2}{4a^2b^2}$, or,

$16{S_{\Delta ABC}}^2 = 4a^2b^2 - (a^2+b^2-c^2)^2 = (2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2) = ((a+b)^2-c^2)(c^2-(a-b)^2)=(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$, or,

$${S_{\Delta ABC}}^2 = p(p-a)(p-b)(p-c)$$

7. Show that

a. $\cos^2\alpha + \cos^2\left(\frac{2\pi}{3}+\alpha\right) + \cos^2\left(\frac{2\pi}{3}-\alpha\right) = \cos^2\alpha + \left(-\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right)^2 + \left(-\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right)^2 = \cos^2\alpha + 2\left(\frac{1}{2}\cos\alpha\right)^2 + 2\left(\frac{\sqrt{3}}{2}\sin\alpha\right)^2 = \frac{3}{2}\cos^2\alpha + \frac{3}{2}\sin^2\alpha = \frac{3}{2}$

b. $\sin\alpha + \sin\left(\frac{2\pi}{3} + \alpha\right) + \sin\left(\frac{4\pi}{3} + \alpha\right) = \sin\alpha + \frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha - \frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha = 0$

c. $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = \frac{3\sin x - 4\sin^3 x}{\sin x} - \frac{4\cos^3 x - 3\cos x}{\cos x} = 6 - 4\sin^2 x - 4\cos^2 x = 2$

8. Without using calculator, find:

a. $\sin 75° = \sin(90° - 15°) = \cos 15° = \cos\frac{30°}{2} = \sqrt{\frac{1}{2}(1 + \cos 30°)} = \sqrt{\frac{2+\sqrt{3}}{4}}$

b. $\cos 75° = \cos(90° - 15°) = \sin 15° = \sin\frac{30°}{2} = \sqrt{\frac{1}{2}(1 - \cos 30°)} = \sqrt{\frac{2-\sqrt{3}}{4}}$

c. $\sin\frac{\pi}{8} = \sin\frac{1}{2}\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}\left(1 - \cos\frac{\pi}{4}\right)} = \sqrt{\frac{2-\sqrt{2}}{4}}$

d. $\cos\frac{\pi}{8} = \cos\frac{1}{2}\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{4}\right)} = \sqrt{\frac{2+\sqrt{2}}{4}}$

e. $\sin\frac{\pi}{16} = \sin\frac{1}{2}\left(\frac{\pi}{8}\right) = \sqrt{\frac{1}{2}\left(1 - \cos\frac{\pi}{8}\right)} = \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}}$

f. $\cos\frac{\pi}{16} = \cos\frac{1}{2}\left(\frac{\pi}{8}\right) = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{8}\right)} = \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{4}}$

g. $\cos\frac{\pi}{2^{n+1}} = \cos\frac{1}{2}\left(\frac{\pi}{2^n}\right) = \sqrt{\frac{1}{2}\left(1 + \cos\frac{\pi}{2^n}\right)} = \sqrt{\frac{2+\sqrt{2+\sqrt{2+\cdots}}}{4}}$

9. Find the sum of the following series,

$$S = \sin x + \sin 3x + \sin 5x + \sin 7x + \cdots + \sin 2017x$$

$$\begin{aligned} 2\sin x\, S &= 2\sin x \sin x + 2\sin x \sin 3x + 2\sin x \sin 5x + \cdots \\ &\quad + 2\sin x \sin 2017x \\ &= 1 - \cos 2x + \cos 2x - \cos 4x + \cos 4x - \cos 6x + \cdots \\ &\quad - \cos 2016x + \cos 2016x - \cos 2018x = 1 - \cos 2018x \end{aligned}$$

$$S = \frac{1 - \cos 2018x}{2\sin x}$$