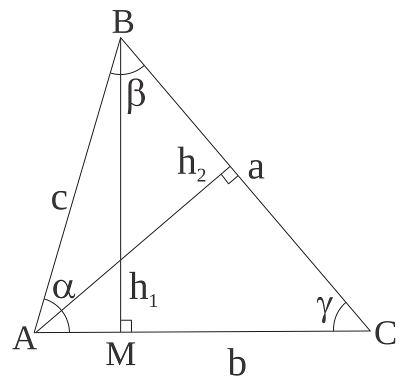
Geometry.

Recap: Geometry of a triangle and trigonometry.

The Law of Sines.

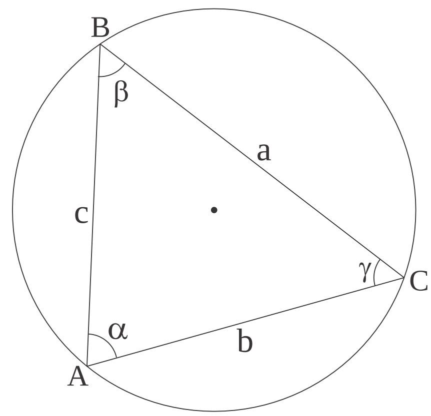


$$c \sin \alpha = h_1 = a \sin \gamma \Rightarrow \frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$$

$$c \sin \beta = h_2 = b \sin \gamma \Rightarrow \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

The Law of Sines generalizes the fact that the greater side lies opposite to the greater angle.

The Extended Law of Sines



$$\sin \alpha = \frac{a}{2R} \Rightarrow \frac{a}{\sin \alpha} = 2R$$

$$\sin \beta = \frac{b}{2R} \Rightarrow \frac{b}{\sin \beta} = 2R$$

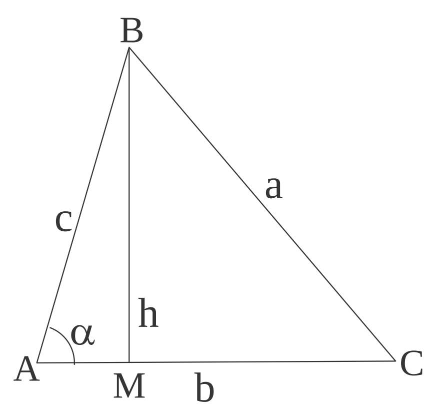
$$\sin \gamma = \frac{c}{2R} \Rightarrow \frac{c}{\sin \gamma} = 2R$$

The Law of Sines states that for any triangle ABC,

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

Where R is the radius of the circumscribed circle.

The Law of Cosines.



For any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

To prove it, we consider right triangles formed by the height AM,

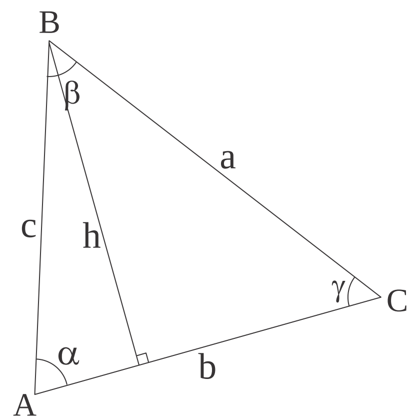
$a^2 = h^2 + |MC|^2$,
$|MC| = b - |AM| = b - c\cos\alpha$,
$h^2 = c^2 - |AM|^2 = c^2 - c^2(\cos\alpha)^2$,

$$a^2 = c^2 - c^2(\cos\alpha)^2 + (b - c\cos\alpha)^2 =$$

$$= c^2 - c^2(\cos\alpha)^2 + b^2 - 2bc\cos\alpha + c^2(\cos\alpha)^2 = b^2 + c^2 - 2bc\cos\alpha$$

Area of a triangle.

Using the Law of Sines in a standard formula, or simply considering the right triangle formed by an altitude opposite to vertex A, one obtains,



$$S_{\Delta ABC} = \frac{1}{2}hb = \frac{1}{2}bc\sin\alpha$$

Similarly, we also can get two more formulas:

$$S_{\Delta ABC} = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ca\sin\beta$$

Using the Law of sines, we also have,

$$S_{\Delta ABC} = \frac{abc}{4R} = 2\,R^2\sin\alpha\sin\beta\sin\gamma$$

where R is the radius of the circumscribed circle. We have also previously shown that

$$S_{\Delta ABC} = \frac{a+b+c}{2}r = sr$$

where r is the radius of the inscribed circle and s the semiperimeter. Finally, the area of the triangle can also be derived from the lengths of the sides by the Heron's formula,

$$S_{\Delta ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

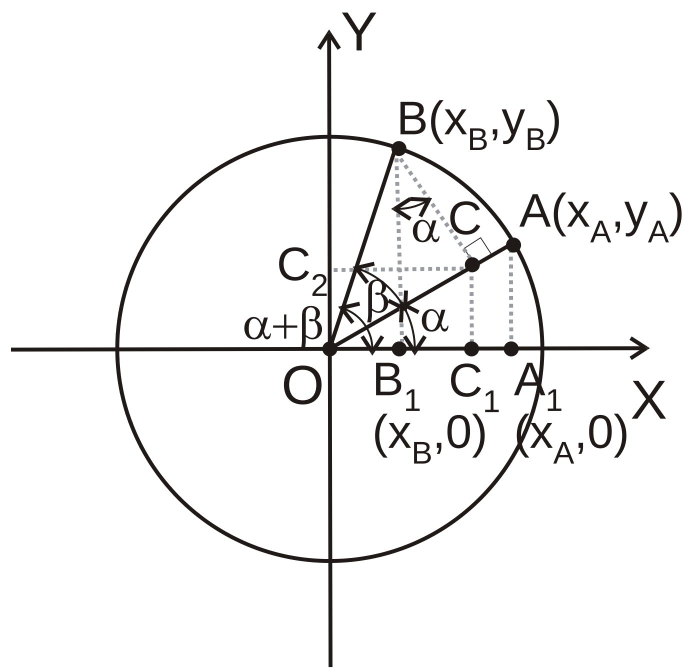
Trigonometry. Trigonometric formulae.

Exercise. Derive expressions for the sine and the cosine of the sum of two angles (see Figure),

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Solution. Consider the circle of a unit radius, $|\boldsymbol{OB}| = |\boldsymbol{OC}| = \mathbf{1}$, in the Figure. Then, $|\boldsymbol{OB_1}| = \mathbf{cos}(\boldsymbol{\alpha}+\boldsymbol{\beta})$, $|\boldsymbol{BB_1}| = \mathbf{sin}(\boldsymbol{\alpha}+\boldsymbol{\beta})$, $|\boldsymbol{OA_1}| = \boldsymbol{x_A} = \mathbf{cos}\,\boldsymbol{\alpha}$, $|\boldsymbol{AA_1}| = \boldsymbol{y_A} = \mathbf{sin}\,\boldsymbol{\alpha}$, etc.

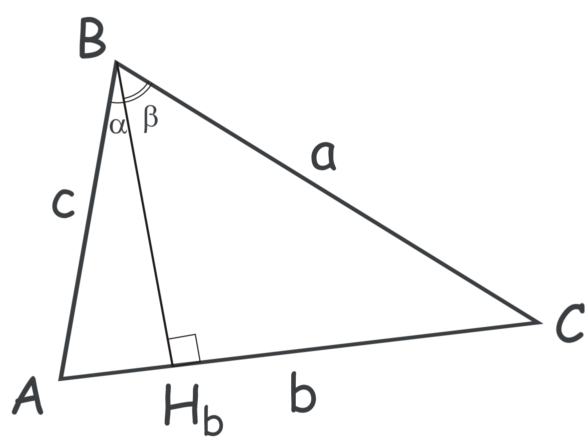


Consequently, $\mathbf{sin}(\boldsymbol{\alpha}+\boldsymbol{\beta}) = |\boldsymbol{BB_1}| = |\boldsymbol{CC_1}| + |\boldsymbol{BC}|\,\mathbf{cos}\,\widehat{\boldsymbol{CBB_1}} = |\boldsymbol{OC}|\,\mathbf{sin}\,\boldsymbol{\alpha} + |\boldsymbol{BC}|\,\mathbf{cos}\,\boldsymbol{\alpha} = \mathbf{sin}\,\boldsymbol{\alpha}\,\mathbf{cos}\,\boldsymbol{\beta} + \mathbf{cos}\,\boldsymbol{\alpha}\,\mathbf{sin}\,\boldsymbol{\beta}$. Similarly, $\mathbf{cos}(\boldsymbol{\alpha}+\boldsymbol{\beta}) = |\boldsymbol{OB_1}| = |\boldsymbol{OC_1}| - |\boldsymbol{B_1C_1}| = |\boldsymbol{OC}|\,\mathbf{cos}\,\boldsymbol{\alpha} - |\boldsymbol{BC}|\mathbf{sin}\,\boldsymbol{\alpha} = \mathbf{cos}\,\boldsymbol{\alpha}\,\mathbf{cos}\,\boldsymbol{\beta} - \mathbf{sin}\,\boldsymbol{\alpha}\,\mathbf{sin}\,\boldsymbol{\beta}$.

Exercise. Derive the addition formulas for sine and cosine using the figure of the triangle with an altitude drawn on the right.

Solution.

1. Consider the area of the triangle ABC,
   $S_{ABC} = \frac{1}{2}ac\sin\widehat{\mathrm{ABC}} = \frac{1}{2}ac\sin(\alpha+\beta)$



On the other hand,

$$S_{ABC} = S_{ABH_b} + S_{BCH_b}$$
$$= \frac{1}{2}|AH_b||BH_b| + \frac{1}{2}|CH_b||BH_b|$$
$$= \frac{1}{2}(c|BH_b|\sin\alpha + a|BH_b|\sin\beta)$$

where $|BH_b| = c\cos\alpha = a\cos\beta$. Substituting this and combining the above equalities, we obtain $\frac{1}{2}ac\sin(\alpha+\beta) = \frac{1}{2}ac(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$.

2. Now let us apply the cosines theorem to the triangle ABC,

$b^2 = a^2 + c^2 - 2ac\cos(\alpha+\beta)$, $\cos(\alpha+\beta) = \frac{a^2+c^2-b^2}{2ac}$, where $b^2 = (c\sin\alpha + a\sin\beta)^2 = c^2\sin^2\alpha + a^2\sin^2\beta + 2ac\sin\alpha\sin\beta$. Combining the two expressions, we obtain,

$$\cos(\alpha+\beta) = \frac{a^2(1-\sin^2\beta) + c^2(1-\sin^2\alpha) - 2ac\sin\alpha\sin\beta}{2ac}$$
$$= \frac{a\cos^2\beta}{2c} + \frac{c\cos^2\alpha}{2a} - \sin\alpha\sin\beta$$

By using $|BH_b| = c\cos\alpha = a\cos\beta$, $\frac{a}{c} = \frac{\cos\alpha}{\cos\beta}$, we obtain,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Exercise. Using the expression for the cosine of the sum of two angles derived above derive the expressions for the sine of the sum of two angles.

Solution. Using the formula for the sine and cosine of the supplementary angle, $\sin\left(\alpha+\frac{\pi}{2}\right) = \cos\alpha$, $\cos\left(\alpha+\frac{\pi}{2}\right) = -\sin\alpha$, and the above expression for $\cos(\alpha+\beta)$ for we obtain,

$$\sin(\alpha+\beta) = -\cos\left(\alpha+\beta+\frac{\pi}{2}\right) = -\left(\cos\alpha\cos\left(\beta+\frac{\pi}{2}\right) - \sin\alpha\sin\left(\beta+\frac{\pi}{2}\right)\right)$$
$$= -\cos\alpha\cdot(-\sin\beta) + \sin\alpha\cos\beta = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Exercise. Using the expressions for the sine and the cosine of the sum of two angles derived above, derive expressions for,

1. $\sin 2\alpha$
2. $\cos 2\alpha$
3. $\sin 3\alpha$
4. $\cos 3\alpha$
5. $\tan(\alpha\pm\beta)$
6. $\cot(\alpha\pm\beta)$
7. $\tan(2\alpha)$
8. $\cot(2\alpha)$