

Algebra.

Polynomials and factorization.

Polynomial is an expression containing variables denoted by some letters, and combined using addition, multiplication and numbers. General form of the n -th degree polynomial of one variable x is,

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x^1 + a_0. \quad (1)$$

This includes quadratic polynomial for $n = 2$, cubic for $n = 3$, etc. The general form for the case of more than one variable is quite complex, for example

$$P_n(x, y) = a_{n,0} x^n + a_{n-1,0} x^{n-1} + \cdots + a_{1,0} x^1 + a_{0,0} + a_{n-1,1} x^{n-1} y + \cdots + a_{1,1} x y + a_{0,1} y + \cdots$$

Please, distinguish variables, such as x and y , which can take any real values, and the coefficients denoted here by a_n , etc, which are just fixed numbers, defining a particular polynomial.

We consider only polynomials with one variable. The number n , which is the highest power of x appearing in the expression of a polynomial P (with non-zero coefficient) is called degree of P and often denoted $\deg(P)$.

One can add, subtract, and multiply polynomials in the obvious way. It is easy to see that for a product of two polynomials, P and Q ,

$$\deg(PQ) = \deg(P) + \deg(Q)$$

However, in general one cannot divide polynomials: expression $\frac{x^3+3}{x^2+x-1}$ is not a polynomial. However, much like with the integers, there is "division with remainder" for polynomials, also known as "long division".

Polynomial division transformation

Theorem. Let $D(x)$ be a polynomial with $\deg(D) > 0$ (i.e., D is not a constant). Then any polynomial $P(x)$ can be uniquely written in the form

$$P(x) = D(x)Q(x) + R(x)$$

where $Q(x)$, $R(x)$ are polynomials, and $\deg(R) < \deg(D)$. The polynomial $R(x)$ is called the remainder upon division of $P(x)$ by $D(x)$.

Polynomial division allows for a polynomial to be written in a divisor-quotient form, which is often advantageous. Consider polynomials $P(x)$, $D(x)$ where $\deg(D) < \deg(P)$. Then, for some quotient polynomial $Q(x)$ and remainder polynomial $R(x)$ with $\deg(R) < \deg(D)$,

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \Leftrightarrow P(x) = D(x)Q(x) + R(x)$$

This rearrangement is known as the **division transformation**, and derives from the corresponding arithmetical identity.

Polynomial long division algorithm for dividing a polynomial by another polynomial of the same or lower degree, is a generalized version of the familiar arithmetic technique called long division. It can be done easily by hand, because it separates an otherwise complex division problem into smaller ones.

Example

Find $\frac{x^3 - 12x^2 - 42}{x - 3}$.

The problem is written like this:

$$\frac{x^3 - 12x^2 + 0x - 42}{x - 3}$$

The quotient and remainder can then be determined as follows:

1. Divide the first term of the numerator by the highest term of the denominator (meaning the one with the highest power of x , which in this case is x). Place the result above the bar ($x^3 \div x = x^2$).

$$\begin{array}{r} x^2 \\ x - 3 \overline{) x^3 - 12x^2 + 0x - 42} \end{array}$$

2. Multiply the denominator by the result just obtained (the first term of the eventual quotient). Write the result under the first two terms of the numerator ($x^2 \cdot (x-3) = x^3 - 3x^2$).

$$\begin{array}{r} x^2 \\ x-3 \overline{) x^3 - 12x^2 + 0x - 42} \\ \underline{x^3 - 3x^2} \end{array}$$

3. Subtract the product just obtained from the appropriate terms of the original numerator (being careful that subtracting something having a minus sign is equivalent to adding something having a plus sign), and write the result underneath ($(x^3 - 12x^2) - (x^3 - 3x^2) = -12x^2 + 3x^2 = -9x^2$). Then, "bring down" the next term from the numerator.

$$\begin{array}{r} x^2 \\ x-3 \overline{) x^3 - 12x^2 + 0x - 42} \\ \underline{x^3 - 3x^2} \\ -9x^2 + 0x \end{array}$$

4. Repeat the previous three steps, except this time use the two terms that have just been written as the numerator.

$$\begin{array}{r} x^2 - 9x \\ x-3 \overline{) x^3 - 12x^2 + 0x - 42} \\ \underline{x^3 - 3x^2} \\ -9x^2 + 0x \\ \underline{-9x^2 + 27x} \\ -27x - 42 \end{array}$$

5. Repeat step 4. This time, there is nothing to "pull down".

$$\begin{array}{r}
 x^2 - 9x - 27 \\
 x - 3 \overline{) x^3 - 12x^2 + 0x - 42} \\
 \underline{x^3 - 3x^2} \\
 -9x^2 + 0x \\
 \underline{-9x^2 + 27x} \\
 -27x - 42 \\
 \underline{-27x + 81} \\
 -123
 \end{array}$$

6. The polynomial above the bar is the quotient, and the number left over (-123) is the remainder.

$$\frac{x^3 - 12x^2 - 42}{x - 3} = \underbrace{x^2 - 9x - 27}_{q(x)} - \underbrace{\frac{123}{x - 3}}_{r(x)/g(x)}$$

The long division algorithm for arithmetic can be viewed as a special case of the above algorithm, in which the variable x is replaced by the specific number 10.

Little Bézout's (polynomial remainder) theorem. Factoring polynomials.

Theorem. The remainder of a polynomial $P(x)$ divided by a linear divisor $(x - a)$ is equal to $P(a)$.

The polynomial remainder theorem follows from the definition of polynomial long division; denoting the divisor, quotient and remainder by, respectively, $G(x)$, $Q(x)$, and $R(x)$, polynomial long division gives a solution of the equation

$$P(x) = Q(x)G(x) + R(x)$$

where the degree of $R(x)$ is less than that of $G(x)$. If we take $G(x) = x - a$ as the divisor, giving the degree of $R(x)$ as 0, i.e. $R(x) = r$,

$$P(x) = Q(x)(x - a) + r. \tag{2}$$

Here r is a number. Setting $x = a$, we obtain $P(a) = r$.

Roots of polynomials.

Definition 1. A number $a \in \mathbb{R}$ is called a **root** of polynomial $P(x)$ if $P(a) = 0$.

Definition 2. A number $a \in \mathbb{R}$ is called a **multiple root** of polynomial $P(x)$ of multiplicity m if $P(x)$ is divisible (without remainder) by $(x - a)^m$ and not divisible by $(x - a)^{m+1}$.

If x_1 is the root of a polynomial $P_n(x)$ of degree n , then $r = 0$, and

$$P_n(x) = (x - x_1)Q_{n-1}(x), \quad (3)$$

where $Q_{n-1}(x)$ is a polynomial of degree $n - 1$. $Q_{n-1}(x)$ is simply the quotient, which can be obtained using the **polynomial long division** (see last class handout). Since x_1 is known to be the root of $P_n(x)$, it follows that the remainder r must be zero.

If we know m roots, $\{x_1, x_2, \dots, x_m\}$, of a polynomial $P_n(x)$ (why is it obvious that $m \leq n$?), then, applying the above reasoning recursively,

$$P_n(x) = (x - x_1)(x - x_2) \dots (x - x_m)Q_{n-m}(x), \quad (4)$$

So if we know that $P_n(x)$ given by (1) has n roots, $\{x_1, x_2, \dots, x_n\}$, then,

$$P_n(x) = a_n(x - x_1)(x - x_2) \dots (x - x_n). \quad (5)$$

If two polynomials,

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x^1 + a_0$$

and

$$Q_n(x) = b_n x^n + b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_2 x^2 + b_1 x^1 + b_0$$

are equal, $P_n(x) = Q_n(x)$, then all corresponding coefficients are equal,

$$a_n = b_n, a_{n-1} = b_{n-1}, a_{n-2} = b_{n-2}, \dots, a_{n-m} = b_{n-m}, \dots, a_1 = b_1, a_0 = b_0. \quad (6)$$

This is the easiest way to obtain the Vieta's theorem and its generalizations for higher-order polynomials.