# MATH 9 <br> HOMEWORK ASSIGNMENT GIVEN ON FEBRUARY 3, 2019. CIRCLE INVERSION 

## Circle inversion

Circle inversion (or simply inversion) is a geometric transformation of the plane. One can think of the inversion as of reflection with respect to a circle which is analogous to a reflection with respect to a straight line. See wikipedia article on Inversive geometry.

## Definition of inversion

Definition. Given a circle $S$ with the center at point $O$ and having a radius $R$ consider the transformation of the plane taking each point $P$ to a point $P^{\prime}$ such that (i) the image $P^{\prime}$ belongs to a ray $O P$ (ii) the distance $\left|O P^{\prime}\right|$ satisfies $|O P| \cdot\left|O P^{\prime}\right|=R^{2}$.


Remark. Strictly speaking the inversion is the transformation not of the whole plane but a plane without point $O$. The point $O$ does not have an image. It is convenient to think of an "infinitely remote" point $O^{\prime}$ added to a euclidian plane so that $O$ and $O^{\prime}$ are mapped to each other by the inversion.

## BASIC PROPERTIES OF INVERSION

We formulated and proved some basic properties of the inversion. The most important properties are that the inversion

- maps circles and straight lines onto circles and straight lines
- preserves angles

These properties are stated as homework problems below. Try to prove them independently. In the following we denote the center of inversion as $O$, the circle of inversion as $S$ and we use prime to denote the images of various objects under inversion.

## Classwork Problems

The properties presented in problems 1-3 are almost obvious.

1. If $P \in S$ then $P^{\prime}=P$. If $P$ is inside $S$, then $P^{\prime}$ is outside $S$. If $P$ is outside $S$, then $P^{\prime}$ is inside $S$.
2. If figure $F$ goes into figure $F^{\prime}$ under inversion then $F^{\prime}$ goes into $F$ under the same inversion. Another way to formulate it is $\left(F^{\prime}\right)^{\prime}=F$.
3. If $l$ is a straight line going through the center of inversion $(O \in l)$ then $l^{\prime}=l$.

The statement of problem 4 is very useful in proving other properties of inversion.
4. Assume that under the inversion with the center $O$ the images of the points $A$ and $B$ are points $A^{\prime}$ and $B^{\prime}$, respectively. Prove that the triangles $O A B$ and $O B^{\prime} A^{\prime}$ are similar.

The properties 5-7 can be formulated as: "the lines and circles are mapped onto lines and circles under inversion". It is convenient to think about lines as of circles of infinite radius. Then we can just say "circles are mapped onto circles".
5. The straight line not going through the center of inversion is mapped onto a circle going through the center of inversion.
6. The circle going through the center of inversion is mapped onto a straight line not going through the center of inversion.
7. The circle not going through the center of inversion is mapped onto a circle not going through the center of inversion.

Remark. Notice that the center of the image of a given circle is NOT the image of the center of the circle. Or in other words, while circles are mapped into circles under the inversions, their centers are not.

The properties 8,9 are almost obvious (but useful!). The problem 9 allows us to transform two circles into parallel lines. This is very useful for solving many geometrical problems.
8. If two circles (or circle and straight line) are tangent at the point $P \neq O$, their images are also tangent (at point $P^{\prime}$ )
9. If two circles (or circle and straight line) are tangent at the center of inversion $O$ their images are two parallel straight lines.

The following problems are to prove that the inversion "preserves angles" (such transformations are called conformal). In math the angle between arbitrary curved lines intersecting at point $P$ is defined as the angle between tangents to those lines at the point $P$.
10. Prove that if a curved line $l_{1}$ intersects the ray $O P$ at point $P$ at angle $\alpha$ the image $l_{1}^{\prime}$ intersects the ray $O P$ at the point $P^{\prime}$ at the same angle.
11. Using the results of the previous problem prove that the angle of intersection of two curves $l_{1}$ and $l_{2}$ is equal to the angle of intersection of their images under inversion $l_{1}^{\prime}$ and $l_{2}^{\prime}$.

Solving the following problems use the properties of inversion described above.
12. Given a circle $S$ and a point $P$ outside the circle construct the image $P^{\prime}$ under an inversion with respect to a circle $S$ using only straightedge and compass.
13. $O$ is the center of inversion, $R$ is the radius of inversion and $P^{\prime}$ and $Q^{\prime}$ are images of points $P$ and $Q$ under inversion, respectively. Express the distance $P^{\prime} Q^{\prime}$ in terms of known distances $P Q, O P, O Q$ and $R$.

Compass and straightedge construction. To construct the inverse $P^{\prime}$ of a point $P$ outside an inversion circle $\varnothing$ :

- Draw the segment from $O$ (center of circle $\varnothing$ ) to $P$.
- Let M be the midpoint of $O P$.
- Draw the circle $c$ with center $M$ going through $P$.
- Let $N$ and $N^{\prime}$ be the points where $\varnothing$ and $c$ intersect.
- Draw segment $N N^{\prime}$.
- $P^{\prime}$ is where $O P$ and $N N^{\prime}$ intersect.



## Homework Problems

1. Given a circle $S$ and a point $P$ inside the circle construct the image $P^{\prime}$ under an inversion with respect to a circle $S$ using only straightedge and compass.
2. Prove that there exists an inversion that maps any pair of non-intersecting circles $S_{1}$ and $S_{2}$ (or any circle and straight line) into the pair of concentric circles.

Using remarkable properties of inversion and compass and straightedge construction one can solve many geometric problem which might be hard to approach otherwise.
3. All possible pairs of mutually tangent circles are inscribed into a given segment (see figure). Find the locus of all tangent points of those circles.

Hint: Consider an inversion with the center $A$.

*4. Ptolemy's theorem states that if states that if $A, B, C$, and $D$ are four points on the circle ordered counterclockwise (or clockwise), the following is correct

$$
A B \cdot C D+B C \cdot D A=A C \cdot B D
$$

Prove this theorem using the circle inversion transformation.
Hint: Consider an inversion with the center $D$. Use the result of the problem 13 from the classwork.
*5. Ptolemy's inequality states that if $A, B, C$, and $D$ are arbitrary four points in the plane, the following is correct

$$
A B \cdot C D+B C \cdot D A \geq A C \cdot B D
$$

Prove this inequality using the method used for the previous problem.
6. Construct a circle going through two given points $A$ and $B$ and tangent to a given circle $S$ (or given straight line $l$ ).
Hint: Consider an inversion with the center $A$.
7. Construct a circle going through a given point $A$ and perpendicular to two given circles $S_{1}$ and $S_{2}$.
Hint: Consider an inversion with the center $A$.
8. Find an inversion that maps a given pair of non-intersecting circles $S_{1}$ and $S_{2}$ (or any circle and straight line) into the pair of concentric circles.
*9. Construct a circle tangent to a given circle $S$ and perpendicular to two given circles $S_{1}$ and $S_{2}$.
Hint: Consider two cases (i) $S_{1}$ intersects $S_{2}$ and (ii) $S_{1}$ does not intersect $S_{2}$. Consider an inversion that (i) maps $S_{1}$ and $S_{2}$ into a pair of straight lines (ii) maps $S_{1}$ and $S_{2}$ into a pair of concentric circles.

## **Problem of Apollonius

Apollonius's problem is to construct circles (using compass and straightedge) that are tangent to three given circles in a plane. One example is shown in the figure.


Apollonius of Perga (ca. 262 BC - ca. 190 BC ) posed and solved this famous problem in his work "Tangencies"; this work has been lost, but a 4th-century report of his results by Pappus of Alexandria has survived. See wikipedia article Problem of Apollonius.

The following problems are directly related to the problem of Apollonius and its solution.
*10. The problem of Apollonius starts with three given circles. How many different cases of mutual configuration of circles can you give? For example, you can have two tangent circle (internally or externally) and one more circle outside of the first two, etc.
*11. Construct a circle tangent to a given pair of parallel lines and a circle (or another line).
*12. Given two circles $S_{1}$ and $S_{2}$ tangent at point $A$ and the third circle $S_{3}$ construct a circle $S_{4}$ (there can be several solutions) tangent to all given circles. (this is a particular case of the problem of Apollonius).
Hint: Consider an inversion with the center $A$.
*13. *Solve the problem of Apollonius.
Idea: Given two circles $S_{1}$ and $S_{2}$ construct circles $\bar{S}_{1}$ and $\bar{S}_{2}$ concentric with $S_{1,2}$ respectively and with radii enlarged by $\frac{1}{2}\left(\left|O_{1} O_{2}\right|-r_{1}-r_{2}\right)$ and the circle $\bar{S}_{3}$ concentric with $S_{3}$ and having radius smaller than $r_{3}$ by the same amount. Solve the problem of Apollonius for $\bar{S}_{1,2,3}$ using the solution of the previous problem and show that the obtained circle $\bar{S}$ with the radius enlarged by the same amount solve the original problem.

