Homework for January 27, 2019.

## Algebra.

Review the classwork handout. Try solving the unsolved problems from the previous homework. Review the classwork exercises and complete those which were not solved (some are repeated below). Solve the following problems.

1. Assume that the set of rational numbers $\mathbb{Q}$ is divided into two subsets, $\mathbb{Q}_{<}$and $\mathbb{Q}_{>}$, such that all elements of $\mathbb{Q}>$ are larger than any element of $\mathbb{Q}_{<}: \forall a \in \mathbb{Q}_{<}, \forall b \in \mathbb{Q}_{>}, a<b$.
a. Prove that if $\mathbb{Q}_{>}$contains the smallest element, $\exists b_{0} \in \mathbb{Q}_{>}, \forall b \in$ $\mathbb{Q}_{>}, b_{0} \leq b$, then $\mathbb{Q}<$ does not contain the largest element
b. Prove that if $\mathbb{Q}_{<}$contains the largest element, $\exists a_{0} \in \mathbb{Q}_{<}, \forall a \in$ $\mathbb{Q}_{<}, a \leq a_{0}$, then $\mathbb{Q}_{>}$does not contain the smallest element
c. Present an example of such a partition, where neither $\mathbb{Q}_{>}$contains the smallest element, nor $\mathbb{Q}_{<}$contains the largest element
2. Prove the following properties of countable sets. For any two countable sets, $A, B$,
a. Union, $A \cup B$, is also countable, $\left(\left(c(A)=\aleph_{0}\right) \wedge\left(c(B)=\aleph_{0}\right)\right)$ $\Rightarrow\left(c(A \cup B)=\aleph_{0}\right)$
b. Product, $A \times B=\{(a, b), a \in A, b \in B\}$, is also countable, $\left(\left(c(A)=\aleph_{0}\right) \wedge\left(c(B)=\aleph_{0}\right)\right) \Rightarrow\left(c(A \cup B)=\aleph_{0}\right)$
c. For a collection of countable sets, $\left\{A_{n}\right\}, c\left(A_{n}\right)=\aleph_{0}$, the union is also countable, $c\left(A_{1} \cup A_{2} \ldots \cup A_{n}\right)=\aleph_{0}$
3. Let $W$ be the set of all "words" that can be written using the alphabet consisitng of 26 lowercase English letters; by a "word", we mean any (finite) sequence of letters, even if it makes no sense - for example, abababaaaaa. Prove that $W$ is countable. [Hint: for any $n$, there are only finitely many words of length $n$.]
4. Compare the following real numbers (are they equal? which is larger?)
a. $1.33333 \ldots=1$. (3) and $4 / 3$
b. $0.09999 \ldots=0.0(9)$ and $1 / 10$
c. $99.9999 \ldots=99 .(9)$ and 100
d. $\sqrt[2]{2}$ and $\sqrt[3]{3}$
5. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
a. $1 / 8$
b. $2 / 7$
c. 0.1
d. $0.33333 \ldots=0 .(3)$
e. $0.13333 \ldots=0.1(3)$
6. Try proving the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

## Ordering and comparison.

1. $\forall a, b \in \mathbb{R}$, one and only one of the following relations holds

- $a=b$
- $a<b$
- $a>b$

2. $\forall a, b \in \mathbb{R}, \exists c \in \mathbb{R},(c>a) \wedge(c<b)$, i.e. $a<c<b$
3. Transitivity. $\forall a, b, c \in \mathbb{R},\{(a<b) \wedge(b<c)\} \Rightarrow(a<c)$
4. Archimedean property. $\forall a, b \in \mathbb{R}, a>b>0, \exists n \in \mathbb{N}$, such that $a<n b$

## Addition and subtraction.

- $\forall a, b \in \mathbb{R}, a+b=b+a$
- $\forall a, b, c \in \mathbb{R},(a+b)+c=a+(b+c)$
- $\forall a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a+0=a$
- $\forall a \in \mathbb{R}, \exists-a \in \mathbb{R}, a+(-a)=0$
- $\forall a, b \in \mathbb{R}, a-b=a+(-b)$
- $\forall a, b, c \in \mathbb{R},(a<b) \Rightarrow(a+c<b+c)$


## Geometry.

Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework.

## Problems.

1. Given two lines, $l$ and $l^{\prime}$, and a point $F$ not on any of those lines, find point $P$ on $l$ such that the (signed) difference of distances from it to $l^{\prime}$ and $F,\left|P^{\prime} L^{\prime}\right|-\left|P^{\prime} F\right|$, is maximal. As seen in the figure, for any $P^{\prime}$ on $l$ the distance to $l^{\prime}$, $\left|P^{\prime} L^{\prime}\right| \leq\left|P^{\prime} L\right| \leq\left|P^{\prime} F\right|+|F L|$, where $|F L|$ is the distance from $F$ to $l^{\prime}$. Hence, $\left|P^{\prime} L^{\prime}\right|-\left|P^{\prime} F\right| \leq|F L|$, and the difference is largest $(=|F L|)$ when point $P$ belongs to the perpendicular $F L$ from point $F$ to $l^{\prime}$.
2. Given line $l$ and points $F_{1}$ and $F_{2}$ lying on different sides of it, find point $P$ on the line $l$ such that the absolute value of the difference in distances from $P$ to points $F_{1}$ and $F_{2}$ is maximal. As above, let $F_{2}{ }^{\prime}$ be the reflection of $F_{2}$ in $l$. Then for any point $X$ on $l,\left|X F_{2}\right|-\left|X F_{1}^{\prime}\right| \leq\left|F_{1} F_{2}^{\prime}\right|$.
3. Find the ( $x, y$ ) coordinates of the common (intersection) point of the two lines, one passing through the origin at
 45 degrees to the $X$-axis, and the other passing through the point $(1,0)$ at 60 degrees to it.
4. Find the ( $x, y$ ) coordinates of the common (intersection) points of the parabola $y=x^{2}$ and of the ellipse centered at the origin and with major axis along the $Y$-axis whose length equals 2 , and the minor axis along the $X$-axis whose length equals 1 .
5. (Skanavi 10.122) Find the locus of the midpoints of all chords of a given circle with the center $O$, which intersect given chord $A B$ of this circle.
6. Three circles of radius $r$ touch each other. Find the area of the triangle $A B C$ formed by tangents to pairs of circles (see figure).
