January 20, 2019

Algebra.

Number sets. Rational and irrational numbers. Real numbers.

Some of the commonly used number sets, such as the set of all digits used in a particular number system, or a set of all permutations of n objects, have finite number of elements, are finite. Others, are infinite. Some of the common number sets that we are familiar with are,

1. The set $\{0,1\}$ of two digits used in the binary number system
2. The set $\{0,1,2,3,4,5,6,7,8,9\}$ of positive integers from 0 to 9 used as the decimal digits
3. The set of all natural integers, which we denote $\mathbb{N}$
4. The set of all integer numbers, which we denote $\mathbb{Z}$
5. The set of all rational numbers, $\frac{m}{n}$, $(\{m,n\} \in \mathbb{Z} \wedge n \neq 0)$, which we denote $\mathbb{Q}$
6. The set of all real numbers, which we denote $\mathbb{R}$
7. The set of all irrational numbers, which we denote D

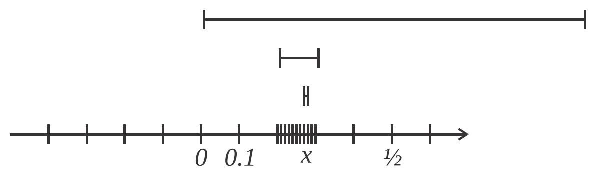
To any non-empty finite set there is a corresponding natural number, the number of elements in this set. Any two sets that have the same number of elements can be related by a bijection, and all such sets form an **equivalence class**, corresponding to this natural number. Thus, natural numbers arise as a characteristic of equivalence classes of finite sets having the same number of elements. Georg Cantor, the originator of set theory, in 1874–1884 extended this concept to infinite sets, where equivalence classes will be characterized by "number of elements" in an infinite set, which is formally infinite.

If elements in the set can be counted by assigning a natural integer to each element, the set is called countable. The set that is not countable is called uncountable. More rigorously, we can give the following definition.

Definition. An infinite set is countable if a bijection exists between this set and the set of natural numbers $\mathbb{N}$.

While rational numbers are defined straightforwardly, as ratios of integers, they are clearly insufficient. It is easy to observe that numbers whose squares are 2, 3, 5, 7, etc., are not rational numbers. Thus, a formal definition for the set of real numbers, $\mathbb{R}$, which includes such irrational numbers, is desirable. First, we note the following important property of rational numbers.

If we represent rational numbers by points on the number line, the resulting **set of the rational points is dense**: there is no interval on the number line, no matter how small, which is free of rational points.



Theorem. Within any interval, no matter how small, there are rational points.

This is easily proven by subdividing the line into arbitrarily small intervals with rational end-points. For instance, starting with an interval with integer endpoints, which contains given small interval $[A, B]$, such that the distance from A to B is a small number ε, we divide the original integer interval into 10 equal parts, and then select the $1/10^{\text{th}}$ segment that contains the interval $[A, B]$. Repeating this procedure successively n times, we will end up with an interval of length ${}^{1}/_{10^n} < \varepsilon$, which will be smaller than the interval $[A, B]$, and therefore will overlap with it. While division into 10 parts corresponds to using the decimal number notation, we could have simply halved the interval each time, or used any other division into rational intervals of length $1/n < \varepsilon$, such that $\exists m, \frac{m}{n} \in [A, B]$.

Corollary. Any interval, no matter how small, contains infinitely many rational points.

Indeed, if there would be a finite number of them, they would divide the interval into a finite number of smaller intervals devoid of rational numbers, which contradicts the above theorem. It would seem from the above that there must be vastly more rational numbers than there are integers: integers are sparse on the number line, there are infinitely many length 1 segments devoid of integers! Surprisingly, though, **the set of the rational numbers is**

countable. This is easily proven by constructing a bijection between the set of natural numbers and the cells of a table which encodes all rational numbers by the position of the cell along the horizontal (numerator) and the vertical (denominator) directions.

Exercise. Complete the argument outlined above proving that the set of rational numbers is countable.

What about the set of real numbers, which contains both rational and irrational numbers, is it countable? First, we have to somehow define the set of real numbers so as to make sure that it does indeed contains all possible irrational numbers. There are a number of different ways to accomplish this.

Real numbers as infinite decimals. Consider decimals from 0 to 1, defined by all possible sequences of digits after the decimal point, $0.a_1a_2a_3 \dots a_n \dots (a_i = 0,1,2, \dots ,9)$. Clearly, all other decimals, lying outside the $[0,1]$ interval, are obtained by simply adding an integer. Some of these decimals denote rational numbers. In particular, such are the sequences with only a finite number of non-zero digits, $\frac{a_1a_2a_3 \dots a_n}{10^n} = 0.a_1a_2a_3 \dots a_n0000 \dots = 0.a_1a_2a_3 \dots a_n(0)$, where (0) denotes periodic sequence of zeros with period 1.

Theorem. Any infinite decimal that ends in a periodically repeating sequence of digits, such as,
$0.a_1a_2a_3 \dots a_n(x_1x_2 \dots x_k) = 0.a_1a_2a_3 \dots a_nx_1x_2 \dots x_kx_1x_2 \dots x_kx_1x_2 \dots x_k \dots$,

represents a rational number,

$$\exists\{p,q\} \in \mathbb{Z}, 0.a_1a_2a_3 \dots a_n(x_1x_2 \dots x_k) = \frac{p}{q}.$$

Proof. This is easily proven by expanding the fraction using the definition of the decimal notation,

$$\begin{aligned}0.a_1a_2a_3 \dots a_n(x_1x_2 \dots x_k) &= \frac{a_1a_2a_3 \dots a_n}{10^n} + \frac{x_1x_2 \dots x_k}{10^{n+k}} + + \frac{x_1x_2 \dots x_k}{10^{n+2k}} + \cdots \\ &= \frac{a_1a_2a_3 \dots a_n}{10^n} + \frac{x_1x_2 \dots x_k}{10^{n+k}}\left(1 + \frac{1}{10^k} + \frac{1}{10^{2k}} + \frac{1}{10^{3k}} + \cdots\right) + \cdots,\end{aligned}$$

which gives an infinite geometric series with the ratio, $q = \frac{1}{10^k}$. For a geometric series with $q < 1$, we can use the following limiting procedure to obtain the sum of an infinite series by extrapolating the known result for the finite geometric series,

$$1 + q + q^2 + \cdots \xleftarrow{n\to\infty} 1 + q + q^2 + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q} \xrightarrow{n\to\infty} \frac{1}{1 - q}.$$

We thus obtain the representation of a cyclic decimal in the form of a rational fraction,

$$0.a_1a_2a_3 \dots a_n(x_1x_2 \dots x_k) = \frac{a_1a_2a_3 \dots a_n}{10^n} + \frac{x_1x_2 \dots x_k}{10^{n+k}} \cdot \frac{1}{1 - \frac{1}{10^k}}$$

$$= \frac{a_1a_2a_3 \dots a_n}{10^n} + \frac{x_1x_2 \dots x_k}{10^n(10^k - 1)} = \frac{(10^k - 1)a_1a_2a_3 \dots a_n + x_1x_2 \dots x_k}{10^{n+k} - 10^n}$$

The opposite statement is also true.

Theorem. Rational numbers that are not finite decimals, are periodic infinite decimals.

Proof. A rational number, $\frac{p}{q}$, is expanded into a decimal fraction by performing the long division by an integer number, q. At each step of this division, there must be a non-zero remainder, otherwise the decimal fraction is finite. However, all possible remainders are integers between 1 and $q - 1$, which means that some remainder, r, has to repeat within at most q divisions. After that, the sequence of remainders between the first and the second appearance of r will repeat periodically, thus yielding a periodic decimal fraction.

Exercise. Show that decimals $0.09999 \dots = 0.0(9)$ and 0.1 represent the same (rational) number.

Infinite decimals that are not periodic represent numbers that are not rational, and therefore are **irrational numbers**.

Real numbers are represented by all possible decimal fractions, both finite and infinite. While this definition of real numbers is quite simple and straightforward, it relies on a particular (decimal) numbers system, and does not offer an equally simple way to port algebraic operations, which have been introduced for rational numbers, to the real numbers.

Real numbers as nested intervals. The construct with a set of nested intervals with the rational endpoints illustrated in the figure above, provides a natural way to define the irrational numbers. This definition is based on a geometrical postulate that an **infinite set of nested intervals whose length tends to zero** (ie is smaller than any arbitrarily chosen small number for all intervals except for a finite subset) **has precisely one point common to all intervals**. This point, even though it is defined by the nested intervals with rational endpoints, itself can be either rational, or irrational. The set of all such points, determined by all possible sets of nested rational intervals, defines all real numbers.

Real numbers as Dedekind cuts. An alternative, axiomatic way to extend rational numbers and define real numbers was proposed by Richard Dedekind in 1872. Let us assume that we can divide the set of rational numbers $\mathbb{Q}$ into two subsets, $\mathbb{Q}_<$ and $\mathbb{Q}_>$, such that all elements of $\mathbb{Q}_>$ are larger than any element of $\mathbb{Q}_<$: $\forall a \in \mathbb{Q}_<, \forall b \in \mathbb{Q}_>, a < b$. The partition of the set of rational numbers into two such subsets is called Dedekind's cut. There are three possibilities in such a partition,

1. $\mathbb{Q}_>$ contains the smallest element, $\exists b_0 \in \mathbb{Q}_>, \forall b \in \mathbb{Q}_>, b_0 < b$
2. $\mathbb{Q}_<$ contains the largest element, $\exists a_0 \in \mathbb{Q}_<, \forall a \in \mathbb{Q}_<, a < a_0$
3. Neither $\mathbb{Q}_>$ contains the smallest element, nor $\mathbb{Q}_<$ contains the largest element

In the third case, where there is neither a largest rational number in $\mathbb{Q}_<$, nor the smallest rational element in $\mathbb{Q}_>$, the cut, according to Dedekind, defines an irrational number. This definition agrees with the definition based on the nested intervals, as each set of nested rational intervals defines a Dedekind cut, if we associate all rational numbers that are larger than the right (larger) side of any of the intervals with $\mathbb{Q}_>$, and the rest of the rational numbers (those to the left of all intervals) with $\mathbb{Q}_<$.

Having defined real numbers as nested intervals, or the Dedekind cuts, it is easy to see that all the usual arithmetic operations and properties of rational numbers are transposed to real numbers. For the case of nested intervals, this is accomplished by applying an operation to the rational endpoints of the two intervals that define two real numbers, and associating the result with the third set of nested intervals so obtained. Similarly, operations for the Dedekind cuts are defined by reference to rational sets that define the cut.

Properties of real numbers. Ordering and comparison.

1. $\forall\, a, b \in \mathbb{R}$, one and only one of the following relations holds
    - $a = b$
    - $a < b$
    - $a > b$
2. $\forall\, a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\, a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\, a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}$, such that $a < nb$
5. Continuity. Consider a set of nested segments $[a_n, b_n], n \in \mathbb{N}, a_n, b_n \in \mathbb{R}$, $a_1 \leq a_2 \leq \cdots \leq a_n \leq b_1 \leq b_2 \leq \cdots b_n$. Then, $\exists\, A, \forall n\; A \in [a_n, b_n]$. If $|a_n - b_n| \to 0$, then such point A is unique.

Properties of real numbers. Addition and subtraction.

- $\forall\, a, b \in \mathbb{R}, a + b = b + a$
- $\forall\, a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
- $\forall\, a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
- $\forall\, a \in \mathbb{R}, \exists -a \in \mathbb{R}, a + (-a) = 0$
- $\forall\, a, b \in \mathbb{R}, a - b = a + (-b)$
- $\forall\, a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$

Properties of real numbers. Multiplication and division.

- $\forall\, a, b \in \mathbb{R}, a \cdot b = b \cdot a$
- $\forall\, a, b, c \in \mathbb{R}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- $\forall\, a, b, c \in \mathbb{R}, (a + b) \cdot c = a \cdot c + b \cdot c$

- $\forall\, a \in \mathbb{R}, \exists 1 \in \mathbb{R}\,, a \cdot 1 = a$
- $\forall\, a \in \mathbb{R}, a \neq 0, \exists \frac{1}{a} \in \mathbb{R}, a \cdot \frac{1}{a} = 1$
- $\forall\, a, b \in \mathbb{R}, b \neq 0, \frac{a}{b} = a \cdot \frac{1}{b}$
- $\forall\, a, b, c \in \mathbb{R}$, c>0, $(a < b) \Rightarrow (a \cdot c < b \cdot c)$
- $\forall\, a \in \mathbb{R}, a \cdot 0 = 0, a \cdot (-1) = -a$