January 13, 2019

Algebra.

Cartesian product.

Given two sets, A and B, we can construct a third set, C, which is made of all possible ordered pairs of the elements of these sets, (a, b), where $a \in A$ and $b \in B$. We thus have a **binary operation**, which acts on a pair of objects (sets A and B) and returns a third object (set C). Following Rene Descartes, who first considered such construction in the context of Cartesian coordinates of points on a plane, in mathematics such operation is called Cartesian product.

A **Cartesian product** is a mathematical operation that returns a (product) set from multiple sets. For two sets A and B, the Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$,

$$A \times B = \{(a, b): a \in A \wedge b \in B\}$$

Example 1. A table can be created from a single row and a single column, by taking the Cartesian product of a set of objects in a row and a set of objects in a column. In the Cartesian product row × column, the cells of the table contain ordered pairs of the form (row object, column object).

Example 2. Another example is a 52 (or 36) card deck. In a 52 card deck, the standard playing card ranks {A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2} form a 13-element set. The card suits {♠, ♥, ♦, ♣} form a four-element set. The Cartesian product of these two sets returns a 52-element set consisting of 52 ordered pairs, which correspond to all 52 possible playing cards. Ranks × Suits returns a set of the form {(A, ♠), (A, ♥), (A, ♦), (A, ♣), (K, ♠), ..., (3, ♣), (2, ♠), (2, ♥), (2, ♦), (2, ♣)}. Suits × Ranks returns a set of the form {(♠, A), (♠, K), (♠, Q), (♠, J), (♠, 10), ..., (♣, 6), (♣, 5), (♣, 4), (♣, 3), (♣, 2)}. Are these two sets different?

The Cartesian product A × B is **not commutative**, because the elements in the ordered pairs are reversed.

$$\{(a, b): a \in A \wedge b \in B\} = A \times B \neq B \times A = \{(b, a): a \in A \wedge b \in B\}$$

Exercise 1. Construct Cartesian product for sets:

- $A = \{13,14\}$; $B = \{1,1\}$
- $A = \{3,5,7\}$; $B = \{7,5,3\}$
- $A = \{a, b, c, d, e, f, g, h\}$; $B = \{1,2,3,4,5,6,7,8\}$
- $A = \{J, F, M, A, M, J, J, A, S, O, N, D\}$; $B = \{n : n \in \mathbb{N} \wedge n \leq 31\}$

Exercise 2. Check non-commutativity for Cartesian product of sets in Exercise 1 (construct $B \times A$).

Exercise 3. For which particular cases is the Cartesian product commutative?

The Cartesian product is **not associative**,

$$(A \times B) \times C \neq A \times (B \times C)$$

For example, if $A = \{1\}$, then,
$(A \times A) \times A = \{ ((1,1),1) \} \neq \{ (1,(1,1)) \} = A \times (A \times A)$

Exercise 4. For which particular cases is the Cartesian product associative?

The Cartesian product has the following property with respect to intersections,

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$$

The above statement is not true if we replace intersection with union,

$$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$$

Exercise 5. Prove the following distributivity properties of Cartesian products,

- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

Equivalence relations and partitions.

Definition. A **binary relation** on a set A,

$$x \sim y, \qquad x, y \in A$$

is a collection of ordered pairs of elements of A, $\{(x, y)\}$, $x, y \in A$. In other words, it is a subset of the Cartesian product $A^2 = A \times A$.

More generally, a binary relation between two sets A and B is a subset of $A \times B$. The terms correspondence, dyadic relation and 2-place relation are synonyms for binary relation.

Example 1. A binary relation $>$ ("is greater than") between real numbers $x, y \in \mathbb{R}$ associates to every real number all real numbers that are to the left of it on the number axis.

Example 2. A binary relation "is the divisor of " between the set of prime numbers P and the set of integers $\mathbb{Z}$ associates every prime p with every integer n that is a multiple of p, but not with integers that are not multiples of p. In this relation, the prime 3 is associated with numbers that include -6, 0, 6, 9, but not 2 or -8; and the prime 5 is associated with numbers that include 0, 10, and 125, but not 6 or 11.

Injections, surjections, bijections between the sets are established by defining the corresponding (injective, surjective, or one-to-one) binary relations between the elements of these sets. A relation $x \sim y$ is,

- **left-total**: $\forall x \in X, \exists y \in Y, x \sim y$, a relation is left-total when it is a function, or a multivalued function;
- **surjective** (right-total, or onto): $\forall y \in Y, \exists x \in X, x \sim y$;
- **injective** (left-unique): $\forall (x_1, x_2, \in X, y \in Y), \big((x_1 \sim y) \wedge (x_2 \sim y) \Rightarrow (x_1 = x_2)\big)$
- **functional** (right-unique, also called univalent, or right-definite): $\forall (x \in X, y_1, y_2, \in Y), \big((x \sim y_1) \wedge (x \sim y_2) \Rightarrow (y_1 = y_2)\big)$, such a binary relation is also called a partial function;
- **one-to-one**: injective and functional.

A binary relation $x \sim y$ is

- **reflexive** if $\forall x \in A$, we have $x \sim x$

- **symmetric** if $\forall x, y \in A$, we have $(x{\sim}y) \Rightarrow (y{\sim}x)$
- **transitive** if $\forall x, y, z \in A$, we have $(x{\sim}y) \wedge (y{\sim}z) \Rightarrow (x{\sim}z)$

Definition. An **equivalence relation** is a binary relation that is reflexive, symmetric, and transitive.

Given an equivalence relation on A, we can define, for every $a \in A$, its **equivalence class** $[a]$ as the following subset of A:

$$[a] = \{x \in A, (x{\sim}a)\}$$

Definition. A **partition** of a set A is decomposition of it into non-intersecting subsets:

$$A = A_1 \cup A_2 \ldots \cup A_n \ldots$$

with $A_i \cap A_j = \emptyset$. It is allowed to have infinitely many subsets A_i.

Theorem. If $\sim$ is an equivalence relation on a set A, then it defines a partition of A into equivalence classes.

Example. Define the equivalence relation on $\mathbb{Z}$ by congruence mod 3: $a \equiv b \; mod \; 3$ if $a - b$ is a multiple of 3. This defines a partition, $[0] = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$, $[1] = \{\ldots, -2, 1, 4, 7, \ldots\}$, $[2] = \{\ldots, -1, 2, 5, 8, \ldots\}$.

Exercise 1. Present examples of binary relations that are, and that are not equivalence relations. For each of the following relations, check whether it is an equivalence relation.

- On the set of all lines in the plane: relation of being parallel
- On the set of all lines in the plane: relation of being perpendicular
- On $\mathbb{R}$: relation given by $x \sim y$ if $x + y \in \mathbb{Z}$
- On $\mathbb{R}$: relation given by $x \sim y$ if $x - y \in \mathbb{Z}$
- On $\mathbb{R}$: relation given by $x \sim y$ if $x > y$
- On $\mathbb{R} - \{0\}$: relation given by $x \sim y$ if $xy > 0$

Exercise 2. Let $\sim$ be an equivalence relation on A.

- Prove that if $a \sim b$, then $[a] = [b]$: $\forall x \in A, x \in [a] \Rightarrow x \in [b]$
- Prove that if $a \nsim b$, then $[a] \cap [b] = \emptyset$.

Exercise 3. Let $f: A \xrightarrow{f} B$ be a function. Define a relation on A by $a \sim b$ if $f(a) = f(b)$. Prove that it is an equivalence relation.

Exercise 4. For a positive integer number $n \in \mathbb{N}$, define relation $\equiv$ on $\mathbb{Z}$ by $a \equiv b$ if $a - b$ is a multiple of n

- Prove that it is an equivalence relation;
- Describe equivalence class $[0]$;
- Prove that equivalence class of $[a + b]$ only depends on equivalence classes of a, b, that is, if $[a] = [a'], [b] = [b']$, then $[a + b] = [a' + b']$.

Exercise 5. Define a relation $\sim$ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ by $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 + x_2 = y_1 + y_2$. Prove that it is an equivalence relation and describe the equivalence class of (1, 2).

Exercise 6. Is it possible to partition the set of all integers, $\mathbb{Z}$, into equivalence classes using the binary relation $p \sim q$: $p \equiv 0 mod(q)$ ("p is a multiple of q"), which was defined in Example 2.