Homework for January 13, 2019.

<u>The homework problems are your opportunity to think and exercise. Some are more challenging than others and you are not expected to solve all of them. However, you are expected to try and solve as many as you can.</u>

Algebra.

Review the classwork handout and complete the exercises that were not solved (some are repeated below). Solve the following problems, including from the previous homework, some of which are repeated below.

1. Using the recurrence relation obtained by solving the old hats problem, derive the formula for a derangement probability,

$$p_n \equiv \frac{!n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

2. For a set A, define the characteristic function χ_A as follows,

$$\chi_A(x) = \begin{cases} 1, if\ x \in A \\ 0, if\ x \notin A \end{cases}$$

Show that χ_A has following properties

$$\chi_A = 1 - \chi_{A'}$$
$$\chi_{A \cap B} = \chi_A \chi_B$$
$$\chi_{A \cup B} = 1 - \chi_{A' \cap B'} = 1 - \chi_{A'} \chi_{B'} = 1 - (1 - \chi_A)(1 - \chi_B)$$
$$= \chi_A + \chi_B - \chi_A \chi_B$$

Write a formula for $\chi_{A \cup B \cup C}$; $\chi_{A \cup B \cup C \cup D}$.

3. Using the inclusion-exclusion principle, find how many natural numbers $n < 1000$ are divisible by 5, 7, 11, or 13.
4. How many passwords of at least 8 characters can one compose using lower and upper case letters and numbers 0 to 9?
5. * If 9 dies are rolled, what is the probability that all 6 numbers appear?
6. * How many permutations of the 26 letters of English alphabet do not contain any of the words *pin*, *fork*, or *rope*?

Geometry.

Review the previous classwork notes. Solve the following problems, including the remaining problems from the previous homework, which are repeated below.

Problems.

1. The expression $d^2 - R^2$ is called the power of point P with respect to a circle of radius R, if $d = |PO|$ is the distance from P to the center O of the circle. The power is positive for points outside the circle; it is negative for points inside the circle, and zero on the circle.
   a. What is the smallest possible value of the power that a point can have with respect to a given circle of radius R? Which point is that?
   b. What is the locus of all points of constant power p (greater than the above minimum) with respect to a given circle? Write the relation between the (x, y) coordinates of such points, assuming that circle is centered at the origin (0,0).
   c. Let t^2 be the power of point P with respect to a circle R. What is the geometrical meaning of it?

2. Review the solution of the radical axis of two circles problem: find the locus of points whose powers with respect to two non-concentric circles are equal. Consider situation when circles are concentric.

3. Complete the following exercises from class. Find the locus of points satisfying each of the following equations or inequalities (graph it on a coordinate plane).

   a. $|x| = |y|$

   b. $|x| + x = |y| + y$

   c. $|x|/x = |y|/y$

   d. $[y] = [x]$

   e. $\{y\} = \{x\}$

   f. $x^2 - y^2 \geq 0$

g. $x^2 + y^2 \leq 1$

h. $x^2 + 8x = 9 - y^2$

4. Describe the locus of all points (x, y) equidistant to the X-axis (i. e. the line $y = 0$) and a given point P $(0,2)$ on the Y-axis. Write the formula relating y and x for these points.
5. (Skanavi 15.105) Find the (x, y) coordinates of the vertex C of an equilateral triangle ABC if A and B have coordinates $A(1,3)$ and $B(3,1)$, respectively.
6. (Skanavi 15.106) Find the (x, y) coordinates of the vertices C and D of a square $ABCD$ if A and B have coordinates $A(2,1)$ and $B(4,0)$, respectively.
7. *Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.
8. Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle.

9. Using the Ptolemy's theorem, prove the following:
    a. Given an equilateral triangle △ABC inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.
    b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.