Homework for January 6, 2018.

The homework problems are your opportunity to think and exercise. Some are more challenging than others and you are not expected to solve all of them. However, you are expected to try and solve as many as you can.

Algebra.

Review the classwork handout and complete the exercises that were not solved (some are repeated below). Solve the following problems, including from the previous homework, some of which are repeated below.

1. Construct bijections between the following sets:
   a. (subsets of the set $\{1, \ldots, n\}$) $\leftrightarrow$ (sequences of zeros and ones of length n)
   b. (5-element subsets of $\{1, \ldots, 15\}$)$\leftrightarrow$ (10-element subsets of $\{1, \ldots, 15\}$)
   c. [set of all ways to put 10 books on two shelves (order on each shelf matters)]$\leftrightarrow$ (set of all ways of writing numbers 1, 2, . . . , 11 in some order) [Hint: use numbers 1. . . 10 for books and 11 to indicate where one shelf ends and the other begins.]
   d. (all integer numbers)$\leftrightarrow$ (all even integer numbers)
   e. (all positive integer numbers)$\leftrightarrow$ (all integer numbers)
   f. (interval (0,1))$\leftrightarrow$ (interval (0,5))
   g. (interval (0,1))$\leftrightarrow$ (halfline $(1, \infty)$) [Hint: try $1/x$.]
   h. (interval (0,1))$\leftrightarrow$ (halfline $(0, \infty)$)
   i. (all positive integer numbers)$\leftrightarrow$ (all integer numbers)
2. Let A be a finite set, with 10 elements. How many bijections f: $A \rightarrow A$ are there? What if A has n elements?
3. Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be given by $f(n) = n^2$ Is this function injective? surjective?
4. Hotel Infinity is a fictional hotel with infinitely many rooms, numbered 1, 2, 3,. . . . Each hotel room is single occupancy: only one guest can stay there at any time.
   a. At some moment, Hotel Infinity is full: all rooms are occupied. Yet, when 2 more guests arrive, the hotel manager says he can give rooms to them, by moving some of the current guests around.

Can you show how? (Hint: Construct a bijection between sets $\{-1, 0, 1, 2, \dots\}$ and N).

b. At some moment, Hotel Infinity is full: all rooms are occupied. Still, the management decides to close half of the rooms — all rooms with odd numbers — for renovation. They claim they can house all their guests in the remaining rooms. Can you show how? (Hint: Construct a bijection between the set of all even positive integers $\{2, 4, 6, \dots\}$ and N).

c. Next to Hotel infinity, a competitor has built Hotel Infinity 2, with infinitely many rooms num- bered by all integers: $\dots, -2, -1, 0, 1, 2, \dots$. Yet, the management of original Hotel Infinity claims that their hotel is no smaller than the competition: they could house all the guests of Hotel Infinity 2 in Hotel Infinity. Could you show how? (Hint: Construct a bijection between the set of all integer numbers $\{\dots, -2, -1, 0, 1, 2, \dots\}$ and N).

5. Using the inclusion-exclusion principle, find the probability that if we randomly write a row of digits from 0 to 9, no digit will appear in its proper ordered position.
6. Among 24 students in a class, 14 study mathematics, 10 study science, and 8 study French. Also, 6 study mathematics and science, 5 study mathematics and French, and 4 study science and French. We know that 3 students study all three subjects. How many of these students study none of the three subjects?
7. In a survey on the students' chewing gum preferences, it was found that
   a. 20 like juicy fruit.
   b. 25 like spearmint.
   c. 33 like watermelon.
   d. 12 like spearmint and juicy fruit.
   e. 16 like juicy fruit and watermelon.
   f. 20 like spearmint and watermelon.
   g. 5 like all three flavors.
   h. 4 like none.

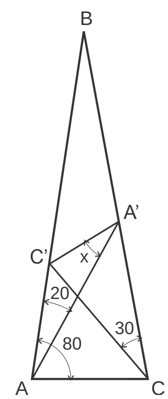
   How many students were surveyed?

8. * If 9 dies are rolled, what is the probability that all 6 numbers appear?
9. * How many permutations of the 26 letters of English alphabet do not contain any of the words *pin*, *fork*, or *rope*?

Geometry.

Review the previous classwork notes. Solve the remaining problems from the previous homework (some are below). Solve the following problems.

Problems.



1. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 30°$ and $\angle BAA' = 20°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.
2. The expression $d^2 - R^2$ is called the power of point P with respect to a circle of radius R, if $d = |PO|$ is the distance from P to the center O of the circle. The power is positive for points outside the circle; it is negative for points inside the circle, and zero on the circle.
   a. What is the smallest possible value of the power that a point can have with respect to a given circle of radius R? Which point is that?
   b. What is the locus of all points of constant power p (greater than the above minimum) with respect to a given circle? Write the relation between the (x, y) coordinates of such points, assuming that circle is centered at the origin (0,0).
   c. Let t^2 be the power of point P with respect to a circle R. What is the geometrical meaning of it?

3. Review the solution of the radical axis of two circles problem: find the locus of points whose powers with respect to two non-concentric circles are equal. Consider situation when circles are concentric.

4. Complete the following exercises from class. Find the locus of points satisfying each of the following equations or inequalities (graph it on a coordinate plane).

   a. $|x| = |y|$

   b. $|x| + x = |y| + y$

   c. $|x|/x = |y|/y$

d. $[y] = [x]$
e. $\{y\} = \{x\}$
f. $x^2 - y^2 \geq 0$
g. $x^2 + y^2 \leq 1$
h. $x^2 + 8x = 9 - y^2$

5. Describe the locus of all points (x, y) equidistant to the X-axis (i. e. the line $y = 0$) and a given point P $(0,2)$ on the Y-axis. Write the formula relating y and x for these points.
6. (Skanavi 15.105) Find the (x, y) coordinates of the vertex C of an equilateral triangle ABC if A and B have coordinates $A(1,3)$ and $B(3,1)$, respectively.
7. (Skanavi 15.106) Find the (x, y) coordinates of the vertices C and D of a square $ABCD$ if A and B have coordinates $A(2,1)$ and $B(4,0)$, respectively.
8. *Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.
9. Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle.

10. Using the Ptolemy's theorem, prove the following:
    a. Given an equilateral triangle $\triangle ABC$ inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.
    b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.