Homework for December 16, 2018.

Algebra.

Review the classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems.

1. Using the inclusion-exclusion principle, find how many natural numbers $n < 100$ are not divisible by 3, 5 or 7.
2. Four letters a, b, c, d, are written down in random order. Using the inclusion-exclusion principle, find probability that at least one letter will occupy its alphabetically ordered place? What is the probability for five letters?
3. Using the inclusion-exclusion principle, find the probability that if we randomly write a row of digits from 0 to 9, no digit will appear in its proper ordered position.
4. Secretary prepared 5 different letters to be sent to 5 different addresses. For each letter, she prepared an envelope with its correct address. If the 5 letters are to be put into the 5 envelopes at random, what is the probability that
    a. no letter will be put into the envelope with its correct address?
    b. only 1 letter will be put into the envelope with its correct address?
    c. only 2 letters will be put into the envelope with its correct address?
    d. only 3 letters will be put into the envelope with its correct address?
    e. only 4 letters will be put into the envelope with its correct address?
    f. all 5 letters will be put into the envelope with its correct address?
5. Among 24 students in a class, 14 study mathematics, 10 study science, and 8 study French. Also, 6 study mathematics and science, 5 study mathematics and French, and 4 study science and French. We know that 3 students study all three subjects. How many of these students study none of the three subjects?
6. In a survey on the students' chewing gum preferences, it was found that
    a. 20 like juicy fruit.
    b. 25 like spearmint.
    c. 33 like watermelon.
    d. 12 like spearmint and juicy fruit.
    e. 16 like juicy fruit and watermelon.
    f. 20 like spearmint and watermelon.

g. 5 like all three flavors.
h. 4 like none.

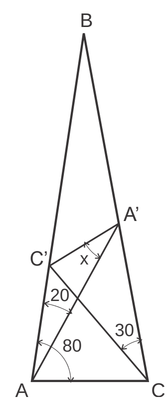
How many students were surveyed?

Geometry.

Review December classwork handouts. Solve the unsolved problems from previous homeworks pasted below, including another version of the "most difficult easy problem".

Problems.

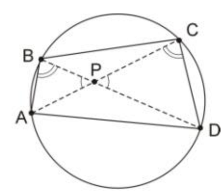
1. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 30°$ and $\angle BAA' = 20°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.

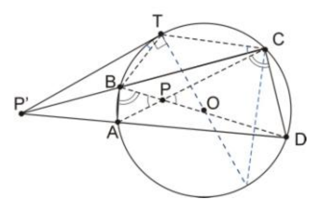


2. Write the proof of the Euclid theorem, which states the following. If two chords AD and BC intersect at a point P' outside the circle, then

$$|P'A||P'D| = |P'B||P'C| = |PT|^2 = d^2 - R^2,$$

where $|PT|$ is a segment tangent to the circle (see Figure).





3. Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle (try using the triangle inequality).

4. Using the Ptolemy's theorem, prove the following:

a. Given an equilateral triangle $\triangle ABC$ inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.

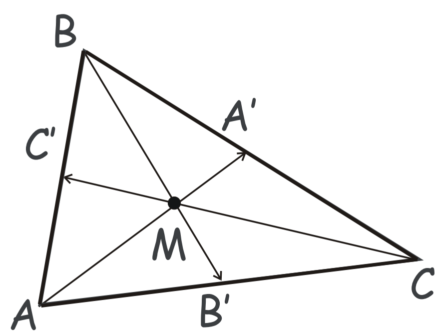
b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.

5. Given a circle of radius R, find the length of the sagitta (Latin for arrow) of the arc AB, which is the perpendicular distance CD from the arc's midpoint (C) to the chord AB across it.
6. Prove the Viviani's theorem:

   The sum of distances of a point P inside an equilateral triangle or on one of its sides, from the sides, equals the length of its altitude. Or, alternately,

   From a point P inside (or on a side) of an equilateral triangle ABC drop perpendiculars PP_a, PP_b, PP_c to its sides. The sum $|PP_a| + |PP_b| + |PP_c|$ is independent of P and is equal to any of the triangle's altitudes.

7. *Three Points are taken at random on an infinite plane. Find the chance of their being the vertices of an obtuse-angled Triangle. Hint: use the Viviani's theorem.
8. In a triangle ABC, Cevian segments AA', BB' and CC' are concurrent and cross at a point M (point C' is on the side AB, point B' is on the side AC, and point A' is on the side BC). Given the ratios $\frac{AC'}{C'B} = p$ and $\frac{AB'}{B'C} = q$, find the ratio $\frac{AM}{MA'}$ (express it through p and q).



9. What is the ratio of the two segments into which a line passing through the vertex A and the middle of the median BB' of the triangle ABC divides the median CC'?
10. In a triangle ABC, A', B' and C' are the tangent points of the inscribed circle and the sides BC, AC, and AB, respectively (see Figure). Prove that cevians AA', BB' and CC' are concurrent (their common point F is called the Gergonne point).